

THE BOREL-MOORE HOMOLOGY OF AN ARITHMETIC QUOTIENT OF THE BRUHAT-TITS BUILDING OF PGL OF A NON-ARCHIMEDEAN LOCAL FIELD IN POSITIVE CHARACTERISTIC AND MODULAR SYMBOLS

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ABSTRACT. We study the homology and the Borel-Moore homology with coefficients in \mathbb{Q} of a quotient (called arithmetic quotient) of the Bruhat-Tits building of PGL of a nonarchimedean local field of positive characteristic by an arithmetic subgroup (a special case of the general definition in Harder's article (Invent. Math. 42, 135-175 (1977))).

We define an analogue of modular symbols in this context and show that the image of the canonical map from homology to Borel-Moore homology is contained in the sub \mathbb{Q} -vector space generated by the modular symbols.

By definition, the limit of the Borel-Moore homology as the arithmetic group becomes small is isomorphic to the space of \mathbb{Q} -valued automorphic forms that satisfy certain conditions at a distinguished (fixed) place (namely that it is fixed by the Iwahori subgroup and the center at the place). We show that the limit of the homology with \mathbb{C} -coefficients is identified with the subspace consisting of cusp forms. We also describe an irreducible subquotient of the limit of Borel-Moore homology as an induced representation in a precise manner and give a multiplicity one type result. modular symbol and arithmetic group and Borel-Moore homology and Bruhat-Tits building and automorphic forms

1. INTRODUCTION

Let us state our result slightly more precisely than in the abstract. Let us give the setup. We let F denote a global field of positive characteristic. Let C be a proper smooth curve over a finite field whose function field is F . Let ∞ be a place of F and let $K = F_\infty$ denote the local field at ∞ .

Let \mathcal{BT}_\bullet be the Bruhat-Tits building for PGL_d of K for a positive integer d . It is a simplicial complex of dimension $d - 1$. Let $\Gamma \subset \mathrm{GL}_d(K)$ be an arithmetic subgroup (see the definition in Section 4.1.1). We consider the homology, the Borel-Moore homology, and the canonical map from homology to Borel-Moore homology of the quotient $\Gamma \backslash \mathcal{BT}_\bullet$.

A building is made of (in particular, a union of) subsimplicial complexes called apartments. These are labeled (not one-to-one) by the set of bases of $K^{\oplus d}$. In the Borel-Moore homology of an apartment, there is defined its fundamental class. When the class corresponds to an F -basis, using the pushforward map for Borel-Moore homology, we obtain a class in the Borel-Moore homology of the quotient $\Gamma \backslash \mathcal{BT}_\bullet$. We regard this class as an analogue of a modular symbol. The first of our main results in its rough form may be stated as follows. See Theorem 10 for the precise form.

Theorem 1. *The image of the canonical injective homomorphism*

$$H_{d-1}(\Gamma \backslash \mathcal{BT}_\bullet, \mathbb{Q}) \rightarrow H_{d-1}^{\mathrm{BM}}(\Gamma \backslash \mathcal{BT}_\bullet, \mathbb{Q})$$

is contained in the subspace generated by the classes of the apartments, which correspond to F -bases.

We refer the reader to the nice introduction in [As-Ru] for a short exposition on the classical modular symbols. In the paper [As-Ru], they consider (the analogue of) the modular symbols for the quotient by some congruence subgroup of $\mathrm{SL}_d(\mathbb{R})$. Our result may be regarded as a non-archimedean analogue of a part of their result, especially Proposition 3.2, p.246.

Let us remark on the difference from the archimedean case and the difficulty in our case. In [As-Ru], they use Borel-Serre bordification (compactification), on which the group Γ acts freely. In the non-archimedean case, there exist several compactifications of (the geometric realization of) the Bruhat-Tits building of PGL (the reader is referred to the introduction in [We1]), but the action is not free. We therefore work with the equivariant (co)homology groups.

There is a reason to use Werner's compactification and not other compactifications. The key in the proof of our result is the explicit construction of the map (4) of Section 5.4 at the level of chain complexes in the indicated direction. We do not have an analogous construction for other compactifications, since for the construction we use the continuous map $s(v_1, \dots, v_d)$ (see Lemma 11) which uses the interpretation of the geometric points of the compactification of the Bruhat-Tits building as the set of semi-norms. A further advantage is that the map $s(v_1, \dots, v_d) \times [g_0, \dots, g_{d-1}]$ readily defines a class in the equivariant homology.

The modular symbols considered here have an application in our other paper [Ko-Ya]. The setup and the statements in Sections 2 and 3 of this article already appeared there. We reproduce them here for convenience. That paper uses the results Lemma 6 and Theorem 10.

We turn to our second result. We let $A = H^0(C \setminus \{\infty\}, \mathcal{O}_C)$. Here we identified a closed point of C and a place of F . We write $\hat{A} = \varprojlim_I A/I$, where the limit is taken over the nonzero ideals of A . We let $\mathbb{A}^\infty = \hat{A} \otimes_A F$ denote the ring of finite adeles. For an open compact subgroup $\mathbb{K} \subset \mathrm{GL}_d(\mathbb{A}^\infty)$, let $X_{\mathbb{K}, \bullet} = \mathrm{GL}_d(F) \backslash (\mathrm{GL}_d(\mathbb{A}^\infty) / \mathbb{K} \times \mathcal{BT}_\bullet)$ (see Section 6.1 for the precise definition). It is the finite disjoint union of spaces of the form $\Gamma \backslash \mathcal{BT}_\bullet$ for some arithmetic subgroup Γ . Then we have the following result. See Proposition 27 and Theorem 28 for the relevant notation.

Theorem 2. (1) *We have*

$$\varinjlim_{\mathbb{K}} H_{d-1}(X_{\mathbb{K}, \bullet}, \mathbb{C}) \cong \bigoplus_{\pi} \pi^\infty$$

as representations of $\mathrm{GL}_d(\mathbb{A})$ where $\pi = \pi^\infty \otimes \pi_\infty$ runs over the irreducible cuspidal automorphic representations of $\mathrm{GL}_d(\mathbb{A})$ such that π_∞ is isomorphic to the Steinberg representation of $\mathrm{GL}_d(K)$.

- (2) *Let $\pi = \pi^\infty \otimes \pi_\infty$ be an irreducible smooth representation of $\mathrm{GL}_d(\mathbb{A}^\infty)$ such that π^∞ appears as a subquotient of $\varinjlim_{\mathbb{K}} H_{d-1}^{\mathrm{BM}}(X_{\mathbb{K}, \bullet}, \mathbb{C})$. Then there exist an integer $r \geq 1$, a partition $d = d_1 + \cdots + d_r$ of d , and irreducible cuspidal automorphic representations π_i of $\mathrm{GL}_{d_i}(\mathbb{A})$ for $i = 1, \dots, r$ which satisfy the following properties:*

- (a) *For each i with $0 \leq i \leq r$, the component $\pi_{i, \infty}$ at ∞ of π_i is isomorphic to the Steinberg representation of $\mathrm{GL}_{d_i}(F_\infty)$.*
- (b) *Let us write $\pi_i = \pi_i^\infty \otimes \pi_{i, \infty}$. Let $P \subset \mathrm{GL}_d$ denote the standard parabolic subgroup corresponding to the partition $d = d_1 + \cdots + d_r$. Then π^∞ is isomorphic to a subquotient of the unnormalized parabolic induction $\mathrm{Ind}_{P(\mathbb{A}^\infty)}^{\mathrm{GL}_d(\mathbb{A}^\infty)} \pi_1^\infty \otimes \cdots \otimes \pi_r^\infty$.*

Moreover for any subquotient H of $\varinjlim_{\mathbb{K}} H_{d-1}^{\mathrm{BM}}(X_{\mathbb{K}, \bullet}, \mathbb{C})$ which is of finite length as a representation of $\mathrm{GL}_d(\mathbb{A}^\infty)$, the multiplicity of π in H is at most one.

Theorem 2 (1) follows from a result of Harder [Har] and the classification theorem due to Mœglin and Waldspurger [Mo-Wa] (see Remark 6.1 for a sketch). We give another proof which does not rely on Harder's result. Theorem 2 (2) without Condition (a) follows from a general theorem of Langlands. In a forthcoming paper by the second author, an application of this result (using Condition (a) and the multiplicity one) will be given. The multiplicity result seems new in that, although we have a restriction that π_∞ is isomorphic to the Steinberg representation, a subquotient of the Borel-Moore homology may not be contained in the discrete part of L^2 -automorphic forms. For the proof of Theorem 2(2), we study the quotient building using the interpretation as the moduli of vector bundles with certain structures on C . The tools that appear are the same as in [Gr] but we actually study the quotient space while in [Gr] they consider only some orbit spaces since they ask only for finite generation of cohomology groups.

One technical problem, which was already addressed by Prasad in the paper by Harder [Har, p.140, Bemerkung], is that the quotient of the Bruhat-Tits building may not be a simplicial complex. In Section 2 we give a generalization of the notion of simplicial complexes so as to include those quotients.

We warn that our use of the term Borel-Moore homology is not a common one. We give some justification in Section 2.2.6.

The paper is organized as follows. In Section 2.1, we consider simplicial complexes. Actually, we generalize the definition of a simplicial complex in the usual sense (which we call strict simplicial complex). One reason for doing so is that while the Bruhat-Tits building itself is canonically a (strict) simplicial complex, the quotient is not so canonically. We also redefine (co)homology in an orientation free manner. This is because the Bruhat-Tits building is not canonically oriented. In Section 3, we recall the definitions of the Bruhat-Tits building and the apartments. Besides the recollections, we give a construction of the fundamental class of an apartment in the Borel-Moore homology group. This serves as an analogue of a cycle (from 0 to $i\infty$ in the upper half plane, for example) in the classical case. In Section 4.1, we give the definition and some properties of an arithmetic group. The first of our main results (Theorem 10) is stated in this section. Section 5 is devoted to the proof of Theorem 10. The key to the proof is the construction of the maps (4) and (5) in Section 5.4.

Sections 6 and 7 are devoted to Theorem 2, and are independent of Section 5. We give the definition of the simplicial complex $X_{\mathbb{K}, \bullet}$ and make precise the relation between the limit of the (Borel-Moore) homology of $X_{\mathbb{K}, \bullet}$ as \mathbb{K} becomes small and the space of (\mathbb{Q} -valued) automorphic forms. The main result of Section 6 is Proposition 33. In Section 7, we prove Theorem 2(2), or Theorem 28. The contents of Sections 7.1 and 7.2 are reformulation of [Gr]. The aim of Section 7.3 is to state Propositions 31 and 33. The proofs are given in Section 7.5 and in Section 7.6 respectively. The proof of Theorem 28 using Propositions 31 and 33 is given in Section 7.4.

2. SIMPLICIAL COMPLEXES AND THEIR (CO)HOMOLOGY

The material of this section (except for the remark in Section 2.2.6) appeared in Sections 3 and 5 of [Ko-Ya]. We collected them for the convenience of readers.

2.1. Simplicial complexes.

2.1.1. Let us recall the notion of (abstract) simplicial complex. A simplicial complex is a pair (Y_0, Δ) of a set Y_0 and a set Δ of finite subsets of Y_0 which satisfies the following conditions:

- If $S \in \Delta$ and $T \subset S$, then $T \in \Delta$.
- If $v \in Y_0$, then $\{v\} \in \Delta$.

In this paper we call a simplicial complex in the sense above a strict simplicial complex, and use the terminology “simplicial complex” in a little broader sense, since we will treat as simplicial complexes some arithmetic quotients of Bruhat-Tits building, in which two different simplices may have the same set of vertices. Bruhat-Tits building itself is a strict simplicial complex. Our primary example of a (nonstrict) simplicial complex is $\Gamma \backslash \mathcal{BT}_\bullet$ for an arithmetic group Γ (to be defined in Section 4.1.2).

We adopt the following definition of a simplicial complex: a simplicial complex is a collection $Y_\bullet = (Y_i)_{i \geq 0}$ of the sets indexed by non-negative integers, equipped with the following additional data

- a subset $V(\sigma) \subset Y_0$ with cardinality $i + 1$, for each $i \geq 0$ and for each $\sigma \in Y_i$ (we call $V(\sigma)$ the set of vertices of σ), and
- an element in Y_j , for each $i \geq j \geq 0$, for each $\sigma \in Y_i$, and for each subset $V' \subset V(\sigma)$ with cardinality $j + 1$ (we denote this element in Y_j by the symbol $\sigma \times_{V(\sigma)} V'$ and call it the face of σ corresponding to V')

which satisfy the following conditions:

- For each $\sigma \in Y_0$, the equality $V(\sigma) = \{\sigma\}$ holds,
- For each $i \geq 0$, for each $\sigma \in Y_i$, and for each non-empty subset $V' \subset V(\sigma)$, the equality $V(\sigma \times_{V(\sigma)} V') = V'$ holds.
- For each $i \geq 0$ and for each $\sigma \in Y_i$, the equality $\sigma \times_{V(\sigma)} V(\sigma) = \sigma$ holds, and
- For each $i \geq 0$, for each $\sigma \in Y_i$, and for each non-empty subsets $V', V'' \subset V(\sigma)$ with $V'' \subset V'$, the equality $(\sigma \times_{V(\sigma)} V') \times_{V'} V'' = \sigma \times_{V(\sigma)} V''$ holds.

Let us call the element of the form $\sigma \times_{V(\sigma)} V'$ for j and V' as above, the j -dimensional face of σ corresponding to V' . We remark here that the symbol $\times_{V(\sigma)}$ does not mean a fiber product in any way.

Any strict simplicial complex gives a simplicial complex in the sense above in the following way. Let (Y_0, Δ) be a strict simplicial complex. We identify Y_0 with the set of subsets of Y_0 with cardinality 1. For $i \geq 1$ let Y_i denote the set of the elements in Δ which has cardinality $i + 1$ as a subset of Y_0 . For $i \geq 1$ and for $\sigma \in Y_i$, we set $V(\sigma) = \sigma$ regarded as a subset of Y_0 . For a non-empty subset $V \subset V(\sigma)$, of cardinality $i' + 1$, we set $\sigma \times_{V(\sigma)} V = V$ regarded as an element of $Y_{i'}$. Then it is easily checked that the collection $Y_\bullet = (Y_i)_{i \geq 0}$ together with the assignments $\sigma \mapsto V(\sigma)$ and $(\sigma, V) \mapsto \sigma \times_{V(\sigma)} V$ forms a simplicial complex.

Let Y_\bullet and Z_\bullet be simplicial complexes. We define a map from Y_\bullet to Z_\bullet to be a collection $f = (f_i)_{i \geq 0}$ of maps $f_i : Y_i \rightarrow Z_i$ of sets which satisfies the following conditions:

- for any $i \geq 0$ and for any $\sigma \in Y_i$, the restriction of f_0 to $V(\sigma)$ is injective and the image of $f|_{V(\sigma)}$ is equal to the set $V(f_i(\sigma))$, and
- for any $i \geq j \geq 0$, for any $\sigma \in Y_i$, and for any non-empty subset $V' \subset V(\sigma)$ with cardinality $j + 1$ we have $f_j(\sigma \times_{V(\sigma)} V') = f_i(\sigma) \times_{V(f_i(\sigma))} f_0(V')$.

2.1.2. There is an alternative, less complicated, equivalent definition of a simplicial complex in the sense above, which we will describe in this paragraph. As it will not be used in this article, the reader may skip this paragraph. For a set S , let $\mathcal{P}^{\text{fin}}(S)$ denote the category whose object are the non-empty finite subsets of S and whose morphisms are the inclusions. Then giving a simplicial complex in our sense is equivalent to giving a pair (Y_0, F) of a set Y_0 and a presheaf F of sets on $\mathcal{P}^{\text{fin}}(Y_0)$ such that $F(\{\sigma\}) = \{\sigma\}$ holds for every $\sigma \in Y_0$. This equivalence is explicitly described as follows: given a simplicial complex Y_\bullet , the corresponding F is the presheaf which associates, to a non-empty finite subset $V \subset Y_0$ with cardinality $i + 1$, the set of elements $\sigma \in Y_i$ satisfying $V(\sigma) = V$.

This alternative definition of a simplicial complex is smarter, nevertheless we have adopted the former definition since it is nearer to the definition of a simplicial complex in the usual sense.

2.2. **Homology and cohomology.** Usually the homology groups of Y_\bullet are defined to be the homology groups of a complex C_\bullet whose component in degree i is the free abelian group generated by the i -simplices of Y_\bullet . For a precise definition of the boundary homomorphism of the complex C_\bullet , we need to choose an orientation of each simplex. In this paper we adopt an alternative, equivalent definition of homology groups which does not require any choice of orientations. The latter definition seems a little complicated at first

glance, however it will soon turn out to be a better way for describing the (co)homology of the arithmetic quotients Bruhat-Tits building, which seems to have no canonical, good choice of orientations.

2.2.1. Orientation. We recall in Sections 2.2.1 and 2.2.2 the precise definitions of the (co)homology, the cohomology with compact support and the Borel-Moore homology of a simplicial complex. When computing (co)homology, one usually fixes an orientation of each simplex once and for all, but we do not. This results in an apparently different definition, but they indeed agree with the usual definition.

We introduce the notion of orientation of a simplex. Let Y_\bullet be a simplicial complex and let $i \geq 0$ be a non-negative integer. For an i -simplex $\sigma \in Y_i$, we let $T(\sigma)$ denote the set of all bijections from the finite set $\{1, \dots, i+1\}$ of cardinality $i+1$ to the set $V(\sigma)$ of vertices of σ . The symmetric group S_{i+1} acts on the set $\{1, \dots, i+1\}$ from the left and hence on the set $T(\sigma)$ from the right. Through this action the set $T(\sigma)$ is a right S_{i+1} -torsor.

We define the set $O(\sigma)$ of orientations of σ to be the $\{\pm 1\}$ -torsor $O(\sigma) = T(\sigma) \times_{S_{i+1}, \text{sgn}} \{\pm 1\}$ which is the push-forward of the S_{i+1} -torsor $T(\sigma)$ with respect to the signature character $\text{sgn} : S_{i+1} \rightarrow \{\pm 1\}$. When $i \geq 1$, the $\{\pm 1\}$ -torsor $O(\sigma)$ is isomorphic, as a set, to the quotient $T(\sigma)/A_{i+1}$ of $T(\sigma)$ by the action of the alternating group $A_{i+1} = \text{Ker sgn} \subset S_{i+1}$. When $i = 0$, the $\{\pm 1\}$ -torsor $O(\sigma)$ is isomorphic to the product $O(\sigma) = T(\sigma) \times \{\pm 1\}$, on which the group $\{\pm 1\}$ acts via its natural action on the second factor.

Let $i \geq 1$ and let $\sigma \in Y_i$. For $v \in V(\sigma)$ let σ_v denote the $(i-1)$ -simplex $\sigma_v = \sigma \times_{V(\sigma)} (V(\sigma) \setminus \{v\})$. There is a canonical map $s_v : O(\sigma) \rightarrow O(\sigma_v)$ of $\{\pm 1\}$ -torsors defined as follows. Let $\nu \in O(\sigma)$ and take a lift $\tilde{\nu} : \{1, \dots, i+1\} \xrightarrow{\cong} V(\sigma)$ of ν in $T(\sigma)$. Let $\tilde{\nu}_v : \{1, \dots, i\} \hookrightarrow \{1, \dots, i+1\}$ denote the unique order-preserving injection whose image is equal to $\{1, \dots, i+1\} \setminus \{\tilde{\nu}^{-1}(v)\}$. It follows from the definition of $\tilde{\nu}_v$ that the composite $\tilde{\nu} \circ \tilde{\nu}_v : \{1, \dots, i\} \rightarrow V(\sigma)$ induces a bijection $\tilde{\nu}_v : \{1, \dots, i\} \xrightarrow{\cong} V(\sigma) \setminus \{v\} = V(\sigma_v)$. We regard $\tilde{\nu}_v$ as an element in $T(\sigma_v)$. We define $s_v : O(\sigma) \rightarrow O(\sigma_v)$ to be the map which sends $\nu \in O(\sigma)$ to $(-1)^{\tilde{\nu}^{-1}(v)}$ times the class of $\tilde{\nu}_v$. It is easy to check that the map s_v is well-defined.

Let $i \geq 2$ and $\sigma \in Y_i$. Let $v, v' \in V(\sigma)$ with $v \neq v'$. We have $(\sigma_v)_{v'} = (\sigma_{v'})_v$. Let us consider the two composites $s_{v'} \circ s_v : O(\sigma) \rightarrow O((\sigma_v)_{v'})$ and $s_v \circ s_{v'} : O(\sigma) \rightarrow O((\sigma_{v'})_v)$. It is easy to check that the equality

$$(2.1) \quad s_{v'} \circ s_v(\nu) = (-1) \cdot s_v \circ s_{v'}(\nu)$$

holds for every $\nu \in O(\sigma)$.

2.2.2. Cohomology and homology. We say that a simplicial complex Y_\bullet is locally finite if for any $i \geq 0$ and for any $\tau \in Y_i$, there exist only finitely many $\sigma \in Y_{i+1}$ such that τ is a face of σ . We recall the four notions of homology or cohomology for a locally finite simplicial complex. Let Y_\bullet be a simplicial complex (resp. a locally finite simplicial complex). For an integer $i \geq 0$, we let $Y'_i = \coprod_{\sigma \in Y_i} O(\sigma)$ and regard it as a $\{\pm 1\}$ -set. Given an abelian group M , we regard the abelian groups $\bigoplus_{\nu \in Y'_i} M$ and $\prod_{\nu \in Y'_i} M$ as $\{\pm 1\}$ -modules in such a way that the component at $\epsilon \cdot \nu$ of $\epsilon \cdot (m_\nu)$ is equal to ϵm_ν for $\epsilon \in \{\pm 1\}$ and for $\nu \in Y'_i$.

For $i \geq 1$, we let $\tilde{\partial}_{i,\oplus} : \bigoplus_{\nu \in Y'_i} M \rightarrow \bigoplus_{\nu \in Y'_{i-1}} M$ (resp. $\tilde{\partial}_{i,\prod} : \prod_{\nu \in Y'_i} M \rightarrow \prod_{\nu \in Y'_{i-1}} M$) denote the homomorphism of abelian groups which sends $m = (m_\nu)_{\nu \in Y'_i}$ to the element $\tilde{\partial}_i(m)$ whose coordinate at $\nu' \in O(\sigma') \subset Y'_{i-1}$ is equal to

$$(2.2) \quad \tilde{\partial}_i(m)_{\nu'} = \sum_{(v,\sigma,\nu)} m_\nu$$

where in the sum in the right hand side (v, σ, ν) runs over the triples of $v \in Y_0 \setminus V(\sigma')$, an element $\sigma \in Y_i$, and $\nu \in O(\sigma)$ which satisfy $V(\sigma) = V(\sigma') \amalg \{v\}$ and $s_v(\nu) = \nu'$. Note that the sum on the right hand side is a finite sum for $\tilde{\partial}_{i,\oplus}$ by definition. One can see also that the sum is a finite sum in the case of $\tilde{\partial}_{i,\prod}$ using the locally finiteness of Y_\bullet . Each of $\tilde{\partial}_{i,\oplus}$ and $\tilde{\partial}_{i,\prod}$ is a homomorphism of $\{\pm 1\}$ -modules. Hence it induces a homomorphism $\partial_{i,\oplus} : (\bigoplus_{\nu \in Y'_i} M)_{\{\pm 1\}} \rightarrow (\bigoplus_{\nu \in Y'_{i-1}} M)_{\{\pm 1\}}$ (resp. $\partial_{i,\prod} : (\prod_{\nu \in Y'_i} M)_{\{\pm 1\}} \rightarrow (\prod_{\nu \in Y'_{i-1}} M)_{\{\pm 1\}}$) of abelian groups, where the subscript $\{\pm 1\}$ means the coinvariants. It follows from the formula (2.1) and the definition of $\partial_{i,\oplus}$ and $\partial_{i,\prod}$ that the family of abelian groups $((\bigoplus_{\nu \in Y'_i} M)_{\{\pm 1\}})_{i \geq 0}$ (resp. $((\prod_{\nu \in Y'_i} M)_{\{\pm 1\}})_{i \geq 0}$) indexed by the integer $i \geq 0$, together with the homomorphisms $\partial_{i,\oplus}$ (resp. $\partial_{i,\prod}$) for $i \geq 1$, forms a complex of abelian groups. The homology groups of this complex are the homology groups $H_*(Y_\bullet, M)$ (resp. the Borel-Moore homology groups $H_*^{\text{BM}}(Y_\bullet, M)$) of Y_\bullet with coefficients in M . We note that there is a canonical map $H_*(Y_\bullet, M) \rightarrow H_*^{\text{BM}}(Y_\bullet, M)$ from homology to Borel-Moore homology induced by the map of complexes $((\bigoplus_{\nu \in Y'_i} M)_{\{\pm 1\}})_{i \geq 0} \rightarrow ((\prod_{\nu \in Y'_i} M)_{\{\pm 1\}})_{i \geq 0}$ given by inclusion at each degree.

The family of abelian groups $(\text{Map}_{\{\pm 1\}}(Y'_i, M))_{i \geq 0}$ (resp. $(\text{Map}_{\{\pm 1\}}^{\text{fin}}(Y'_i, M))_{i \geq 0}$ where the superscript fin means finite support) of the $\{\pm 1\}$ -equivariant maps from Y'_i to M forms a complex of abelian groups in a similar manner. (One uses the locally finiteness of Y_\bullet for the latter.) The cohomology groups of this complex

are the cohomology groups $H^*(Y_\bullet, M)$ (resp. the cohomology groups with compact support $H_c^*(Y_\bullet, M)$) of Y_\bullet with coefficients in M .

2.2.3. Universal coefficient theorem. It follows from the definition that the following universal coefficient theorem holds. That is, for a simplicial complex Y_\bullet , there exist canonical short exact sequences

$$0 \rightarrow H_i(Y_\bullet, \mathbb{Z}) \otimes M \rightarrow H_i(Y_\bullet, M) \rightarrow \text{Tor}_1^{\mathbb{Z}}(H_{i-1}(Y_\bullet, \mathbb{Z}), M) \rightarrow 0$$

and

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_{i-1}(Y_\bullet, \mathbb{Z}), M) \rightarrow H^i(Y_\bullet, M) \rightarrow \text{Hom}_{\mathbb{Z}}(H_i(Y_\bullet, \mathbb{Z}), M) \rightarrow 0.$$

for any abelian group M .

Similarly, for a locally finite simplicial complex Y_\bullet , we have short exact sequences

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_c^{i+1}(Y_\bullet, \mathbb{Z}), M) \rightarrow H_i^{\text{BM}}(Y_\bullet, M) \rightarrow \text{Hom}_{\mathbb{Z}}(H_c^i(Y_\bullet, \mathbb{Z}), M) \rightarrow 0$$

and

$$0 \rightarrow H_c^i(Y_\bullet, \mathbb{Z}) \otimes M \rightarrow H_c^i(Y_\bullet, M) \rightarrow \text{Tor}_1^{\mathbb{Z}}(H_c^{i+1}(Y_\bullet, \mathbb{Z}), M) \rightarrow 0$$

for any abelian group M . The canonical inclusions

$$\left(\bigoplus_{\nu \in Y'_i} M \right)_{\{\pm 1\}} \hookrightarrow \left(\prod_{\nu \in Y'_i} M \right)_{\{\pm 1\}} \text{ and } \text{Map}_{\{\pm 1\}}^{\text{fin}}(Y'_i, M) \hookrightarrow \text{Map}_{\{\pm 1\}}(Y'_i, M)$$

for $i \geq 0$ induce homomorphisms $H_i(Y_\bullet, M) \rightarrow H_i^{\text{BM}}(Y_\bullet, M)$ and $H_c^i(Y_\bullet, M) \rightarrow H^i(Y_\bullet, M)$ of abelian groups, respectively.

2.2.4. Let $f = (f_i)_{i \geq 0} : Y_\bullet \rightarrow Z_\bullet$ be a map of simplicial complexes. For each integer $i \geq 0$ and for each abelian group M , the map f canonically induces homomorphisms $f_* : H_i(Y_\bullet, M) \rightarrow H_i(Z_\bullet, M)$ and $f^* : H^i(Z_\bullet, M) \rightarrow H^i(Y_\bullet, M)$. We say that the map f is finite if the subset $f_i^{-1}(\sigma)$ of Y_i is finite for any $i \geq 0$ and for any $\sigma \in Z_i$. If Y_\bullet and Z_\bullet are locally finite, and if f is finite, then f canonically induces the pushforward homomorphism $f_* : H_i^{\text{BM}}(Y_\bullet, M) \rightarrow H_i^{\text{BM}}(Z_\bullet, M)$ and the pullback homomorphism $f^* : H_c^i(Z_\bullet, M) \rightarrow H_c^i(Y_\bullet, M)$.

2.2.5. Let Y_\bullet be a simplicial complex. We associate a CW complex $|Y_\bullet|$ which we call the geometric realization of Y_\bullet . Let $I(Y_\bullet)$ denote the disjoint union $\coprod_{i \geq 0} Y_i$. We define a partial order on the set $I(Y_\bullet)$ by saying that $\tau \leq \sigma$ if and only if τ is a face of σ . For $\sigma \in I(Y_\bullet)$, we let Δ_σ denote the set of maps $f : V(\sigma) \rightarrow \mathbb{R}_{\geq 0}$ satisfying $\sum_{v \in V(\sigma)} f(v) = 1$. We regard Δ_σ as a topological space whose topology is induced from that of the real vector space $\text{Map}(V(\sigma), \mathbb{R})$. If τ is a face of σ , we regard the space Δ_τ as the closed subspace of Δ_σ which consists of the maps $V(\sigma) \rightarrow \mathbb{R}_{\geq 0}$ whose support is contained in the subset $V(\tau) \subset V(\sigma)$. We let $|Y_\bullet|$ denote the colimit $\varinjlim_{\sigma \in I(Y_\bullet)} \Delta_\sigma$ in the category of topological spaces and call it the geometric realization of Y_\bullet . It follows from the definition that the geometric realization $|Y_\bullet|$ has a canonical structure of CW-complex.

2.2.6. Cellular versus singular. We give a remark on the use of the term ‘‘Borel-Moore homology’’ in this paragraph. Given a strict simplicial complex, its cohomology, homology and cohomology with compact support (for a locally finite strict simplicial complex) are usually defined as above, and called cellular (co)homology. See for example [Hatc].

On the other hand, there is also the singular (co)homology and with compact support that are defined using the singular (co)chain complex. It is well-known that the cellular (co)homology groups (with compact support) are isomorphic to the singular (co)homology groups (with compact support) of the geometric realization. The same proof applies to the generalized simplicial complexes and gives an isomorphism between the cellular and the singular theories.

For the Borel-Moore homology, we did not find a cellular definition as above, except in Hattori [Hatt] where he does not call it the Borel-Moore homology. He also gives a definition using singular chains and shows that the two homology groups are isomorphic.

There are several definitions of Borel-Moore homology that may be associated to a (strict) simplicial complex. The definition of the Borel-Moore homology for PL manifolds is found in Haefliger [Ha]. There is also a sheaf theoretic definition in Iversen [Iv]. More importantly, there is the general definition which concerns the intersection homology. However, we did not find a statement in the literature and we did not check that the cellular definition in Hattori (same as the one given in this article) is isomorphic to the other Borel-Moore homology theories.

3. THE BRUHAT-TITS BUILDING AND APARTMENTS

For the general theory of Bruhat-Tits building and apartments, the reader is referred to the book [Ab-Br].

3.1. The Bruhat-Tits building of PGL_d . In the following paragraphs, we recall the definition of the Bruhat-Tits building of PGL_d over K . We recall that it is a strict simplicial complex.

3.1.1. Notation. Let K be a nonarchimedean local field. We let $\mathcal{O} \subset K$ denote the ring of integers. We fix a uniformizer $\varpi \in \mathcal{O}$. Let $d \geq 1$ be an integer. Let $V = K^{\oplus d}$. We regard it as the set of row vectors so that $\mathrm{GL}_d(K)$ acts from the right by multiplication.

3.1.2. Bruhat-Tits building ([Br-Ti]). An \mathcal{O} -lattice in V is a free \mathcal{O} -submodule of V of rank d . We denote by $\mathrm{Lat}_{\mathcal{O}}(V)$ the set of \mathcal{O} -lattices in V . We regard the set $\mathrm{Lat}_{\mathcal{O}}(V)$ as a partially ordered set whose elements are ordered by the inclusions of \mathcal{O} -lattices. Two \mathcal{O} -lattices L, L' of V are called homothetic if $L = \varpi^j L'$ for some $j \in \mathbb{Z}$. Let $\overline{\mathrm{Lat}}_{\mathcal{O}}(V)$ denote the set of homothety classes of \mathcal{O} -lattices V . We denote by cl the canonical surjection $\mathrm{cl} : \mathrm{Lat}_{\mathcal{O}}(V) \rightarrow \overline{\mathrm{Lat}}_{\mathcal{O}}(V)$. We say that a subset S of $\overline{\mathrm{Lat}}_{\mathcal{O}}(V)$ is totally ordered if $\mathrm{cl}^{-1}(S)$ is a totally ordered subset of $\mathrm{Lat}_{\mathcal{O}}(V)$. The pair $(\overline{\mathrm{Lat}}_{\mathcal{O}}(V), \Delta)$ of the set $\overline{\mathrm{Lat}}_{\mathcal{O}}(V)$ and the set Δ of totally ordered finite subsets of $\overline{\mathrm{Lat}}_{\mathcal{O}}(V)$ forms a strict simplicial complex. The Bruhat-Tits building of PGL_d over K is a simplicial complex \mathcal{BT}_{\bullet} which is isomorphic to the simplicial complex associated to this strict simplicial complex. In the next paragraphs we explicitly describe the simplicial complex \mathcal{BT}_{\bullet} .

3.1.3. For an integer $i \geq 0$, let $\widetilde{\mathcal{BT}}_i$ be the set of sequences $(L_j)_{j \in \mathbb{Z}}$ of \mathcal{O} -lattices in V indexed by $j \in \mathbb{Z}$ such that $L_j \supsetneq L_{j+1}$ and $\varpi L_j = L_{j+i+1}$ hold for all $j \in \mathbb{Z}$. In particular, if $(L_j)_{j \in \mathbb{Z}}$ is an element in $\widetilde{\mathcal{BT}}_0$, then $L_j = \varpi^j L_0$ for all $j \in \mathbb{Z}$. We identify the set \mathcal{BT}_0 with the set $\mathrm{Lat}_{\mathcal{O}}(V)$ by associating the \mathcal{O} -lattice L_0 to an element $(L_j)_{j \in \mathbb{Z}}$ in \mathcal{BT}_0 . We say that two elements $(L_j)_{j \in \mathbb{Z}}$ and $(L'_j)_{j \in \mathbb{Z}}$ in $\widetilde{\mathcal{BT}}_i$ are equivalent if there exists an integer ℓ satisfying $L'_j = L_{j+\ell}$ for all $j \in \mathbb{Z}$. We denote by \mathcal{BT}_i the set of the equivalence classes in $\widetilde{\mathcal{BT}}_i$. For $i = 0$, the identification $\widetilde{\mathcal{BT}}_0 \cong \mathrm{Lat}_{\mathcal{O}}(V)$ gives an identification $\mathcal{BT}_0 \cong \overline{\mathrm{Lat}}_{\mathcal{O}}(V)$.

Let $\sigma \in \mathcal{BT}_i$ and take a representative $(L_j)_{j \in \mathbb{Z}}$ of σ . For $j \in \mathbb{Z}$, let us consider the class $\mathrm{cl}(L_j)$ in $\overline{\mathrm{Lat}}_{\mathcal{O}}(V)$. Since $\varpi L_j = L_{j+i+1}$, we have $\mathrm{cl}(L_j) = \mathrm{cl}(L_{j+i+1})$. Since $L_j \supsetneq L_k \supsetneq \varpi L_j$ for $0 \leq j < k \leq i$, the elements $\mathrm{cl}(L_0), \dots, \mathrm{cl}(L_i) \in \overline{\mathrm{Lat}}_{\mathcal{O}}(V)$ are distinct. Hence the subset $V(\sigma) = \{\mathrm{cl}(L_j) \mid j \in \mathbb{Z}\} \subset \mathcal{BT}_0$ has cardinality $i+1$ and does not depend on the choice of $(L_j)_{j \in \mathbb{Z}}$. It is easy to check that the map from \mathcal{BT}_i to the set of finite subsets of $\overline{\mathrm{Lat}}_{\mathcal{O}}(V)$ which sends $\sigma \in \mathcal{BT}_i$ to $V(\sigma)$ is injective and that the set $\{V(\sigma) \mid \sigma \in \mathcal{BT}_i\}$ is equal to the set of totally ordered subsets of $\overline{\mathrm{Lat}}_{\mathcal{O}}(V)$ with cardinality $i+1$. In particular, for any $j \in \{0, \dots, i\}$ and for any subset $V' \subset V(\sigma)$ of cardinality $j+1$, there exists a unique element in \mathcal{BT}_j , which we denote by $\sigma \times_{V(\sigma)} V'$, such that $V(\sigma \times_{V(\sigma)} V')$ is equal to V' . Thus the collection $\mathcal{BT}_{\bullet} = \coprod_{i \geq 0} \mathcal{BT}_i$ together with the data $V(\sigma)$ and $\sigma \times_{V(\sigma)} V'$ forms a simplicial complex which is canonically isomorphic to the simplicial complex associated to the strict simplicial complex $(\overline{\mathrm{Lat}}_{\mathcal{O}}(V), \Delta)$ which we introduced in the first paragraph of Section 3.1.2. We call the simplicial complex \mathcal{BT}_{\bullet} the Bruhat-Tits building of PGL_d over K .

3.1.4. The simplicial complex \mathcal{BT}_{\bullet} is of dimension at most $d-1$, by which we mean that \mathcal{BT}_i is an empty set for $i > d-1$. It follows from the fact that $\widetilde{\mathcal{BT}}_i$ is an empty set for $i > d-1$, which we can check as follows. Let $i > d-1$ and assume that there exists an element $(L_j)_{j \in \mathbb{Z}}$ in $\widetilde{\mathcal{BT}}_i$. Then for $j = 0, \dots, i+1$, the quotient L_j/L_{i+1} is a subspace of the d -dimensional $(\mathcal{O}/\varpi\mathcal{O})$ -vector space $L_0/L_{i+1} = L_0/\varpi L_0$. These subspaces must satisfy $L_0/L_{i+1} \supsetneq L_1/L_{i+1} \supsetneq \dots \supsetneq L_{i+1}/L_{i+1}$. It is impossible since $i+1 > d$.

3.2. Apartments.

Here we recall the definition of the apartments which are simplicial subcomplexes of the Bruhat-Tits building. We then associate to each apartment a class in the Borel-Moore homology of a quotient of the Bruhat-Tits building. This class is an analogue of a modular symbol.

3.2.1. We give an explicit description of the simplicial complex A_{\bullet} below without making use of the theory of root systems. For the viewpoint in the general theory of root systems, we refer the reader to [Ab-Br, p. 523, 10.1.7 Example].

set $A_0 = \mathbb{Z}^{\oplus d}/\mathbb{Z}(1, \dots, 1)$. For two elements $x = (x_j), y = (y_j) \in \mathbb{Z}^{\oplus d}$, we write $x \leq y$ when $x_j \leq y_j$ for all $1 \leq j \leq d$. We say that a subset $\tilde{\sigma} \subset \mathbb{Z}^{\oplus d}$ is small if for any two elements $x, y \in \tilde{\sigma}$ we have either $x \leq y \leq x + (1, \dots, 1)$ or $y \leq x \leq y + (1, \dots, 1)$. Explicitly, this means that $\tilde{\sigma}$ is a finite set and is of the form $\tilde{\sigma} = \{x_0, \dots, x_i\}$ for some elements x_0, \dots, x_i satisfying $x_0 \leq \dots \leq x_i \leq x_{i+1} = x_0 + (1, \dots, 1)$. We say that a finite subset $\sigma \subset A_0$ has a small lift to $\mathbb{Z}^{\oplus d}$ if there exists a small subset $\tilde{\sigma} \subset \mathbb{Z}^{\oplus d}$ which maps bijectively onto σ under the canonical surjection $\mathbb{Z}^{\oplus d} \twoheadrightarrow A_0$. For $i \geq 0$, we let A_i denote the set of the subsets $\sigma \subset A_0$ with cardinality $i+1$ which has a small lift to $\mathbb{Z}^{\oplus d}$. It is clear that the pair $(A_0, \coprod_{i \geq 0} A_i)$ forms a strict simplicial complex and the collection $A_{\bullet} = (A_i)_{i \geq 0}$ is the simplicial complex associated to the strict simplicial complex $(A_0, \coprod_{i \geq 0} A_i)$. We note that A_i is an empty set for $i \geq d$, since by definition there is no small subset of $\mathbb{Z}^{\oplus d}$ with cardinality larger than d .

3.2.2. Let v_1, \dots, v_d be a basis of $V = K^{\oplus d}$. We define a map $\iota_{v_1, \dots, v_d} : A_{\bullet} \rightarrow \mathcal{BT}_{\bullet}$ of simplicial complexes.

Let $\tilde{\iota}_{v_1, \dots, v_d} : \mathbb{Z}^{\oplus d} \rightarrow \widetilde{\mathcal{BT}}_0$ denote the map which sends the element $(n_1, \dots, n_d) \in \mathbb{Z}^d$ to the \mathcal{O} -lattice $\mathcal{O}\varpi^{n_1}v_1 \oplus \mathcal{O}\varpi^{n_2}v_2 \oplus \dots \oplus \mathcal{O}\varpi^{n_d}v_d$. Let $i \geq 0$ be an integer and let $\sigma \in A_i$. Take a small subset $\tilde{\sigma} \subset \mathbb{Z}^{\oplus d}$ with cardinality $i+1$ which maps bijectively onto σ under the surjection $\mathbb{Z}^{\oplus d} \rightarrow \mathbb{Z}^{\oplus d}/\mathbb{Z}(1, \dots, 1) = A_0$. By definition the set $\tilde{\sigma}$ is of the form $\tilde{\sigma} = \{x_0, \dots, x_i\}$ where $x_0, \dots, x_i \in \mathbb{Z}^{\oplus d}$ satisfy $x_0 \leq \dots \leq x_i \leq x_{i+1}$ where we have set $x_{i+1} = x_0 + (1, \dots, 1)$. For each integer $j \in \mathbb{Z}$ we write j in the form $j = m(i+1) + r$ with $m \in \mathbb{Z}$ and $r \in \{0, \dots, i\}$, and set $x_j = x_r + m(1, \dots, 1)$ and $L_j = \tilde{\iota}_{v_1, \dots, v_d}(x_j)$. The sequence $(L_j)_{j \in \mathbb{Z}}$ of \mathcal{O} -lattices gives an element $\tilde{\iota}_{v_1, \dots, v_d, i}(\tilde{\sigma})$ in $\widetilde{\mathcal{BT}}_i$. We denote by $\iota_{v_1, \dots, v_d, i}(\sigma)$ the class of $\tilde{\iota}_{v_1, \dots, v_d, i}(\tilde{\sigma})$ in \mathcal{BT}_i .

Lemma 3. *The class $\iota_{v_1, \dots, v_d, i}(\sigma)$ does not depend on the choice of a small lift $\tilde{\sigma}$.*

Proof. The inverse image of σ under the canonical surjection $\mathbb{Z}^{\oplus d} \rightarrow \mathbb{Z}^{\oplus d}/\mathbb{Z}(1, \dots, 1)$ is equal to $\{x_j \mid j \in \mathbb{Z}\}$. Since $x_j \leq x_{j'}$ for $j \leq j'$ and $x_{j+i+1} = x_j + (1, \dots, 1)$, any small subset $\tilde{\sigma}'$ of $\mathbb{Z}^{\oplus d}$ with cardinality $i+1$ which maps bijectively onto σ is of the form $\tilde{\sigma}' = \{x_l, x_{l+1}, \dots, x_{l+i}\}$ for some $l \in \mathbb{Z}$. The element $\tilde{\iota}_{v_1, \dots, v_d, i}(\tilde{\sigma}')$ is the sequence $(L'_j)_{j \in \mathbb{Z}}$, where $L'_j = L_{j+l}$. Hence the two elements $\tilde{\iota}_{v_1, \dots, v_d, i}(\tilde{\sigma})$ and $\tilde{\iota}_{v_1, \dots, v_d, i}(\tilde{\sigma}')$ give the same element in \mathcal{BT}_i . \square

It is easy to check that the map $\iota_{v_1, \dots, v_d, i} : A_i \rightarrow \mathcal{BT}_i$ is injective for every $i \geq 0$ and that the collection of the map $\iota_{v_1, \dots, v_d, i}$ forms a map $\iota_{v_1, \dots, v_d} : A_{\bullet} \rightarrow \mathcal{BT}_{\bullet}$ of simplicial complexes. We define a simplicial subcomplex $A_{v_1, \dots, v_d, \bullet}$ of \mathcal{BT}_{\bullet} to be the image of the map ι_{v_1, \dots, v_d} so that $A_{v_1, \dots, v_d, i}$ is the image of the map $\iota_{v_1, \dots, v_d, i}$ for each $i \geq 0$. We call the subcomplex $A_{v_1, \dots, v_d, \bullet}$ of \mathcal{BT}_{\bullet} the apartment in \mathcal{BT}_{\bullet} corresponding to the basis v_1, \dots, v_d . Since the map $\iota_{v_1, \dots, v_d, i}$ is injective for every $i \geq 0$, the map ι_{v_1, \dots, v_d} induces an isomorphism $A_{\bullet} \xrightarrow{\cong} A_{v_1, \dots, v_d, \bullet}$ of simplicial complexes.

3.2.3. We introduce a special element β in the group $H_{d-1}^{\text{BM}}(A_{\bullet}, \mathbb{Z})$, which is an analogue of the fundamental class. Let $\sigma \in A_{d-1}$ and take a small lift $\tilde{\sigma} \subset \mathbb{Z}^{\oplus d}$ to $\mathbb{Z}^{\oplus d}$. By definition the set $\tilde{\sigma}$ is of the form $\tilde{\sigma} = \{x_1, \dots, x_d\}$ with $x_0 \leq x_1 \leq \dots \leq x_d$ where we have set $x_0 = x_d - (1, \dots, 1)$. It follows from this property that for each integer i with $1 \leq i \leq d$ there exists a unique integer $w(i)$ with $1 \leq w(i) \leq d$ such that the $w(i)$ -th coordinate of $x_i - x_{i-1}$ is equal to 1 and the other coordinates of $x_i - x_{i-1}$ are equal to zero. Since we have $\sum_{i=1}^d (x_i - x_{i-1}) = x_d - x_0 = (1, \dots, 1)$, the map $w : \{1, \dots, d\} \rightarrow \{1, \dots, d\}$ is injective. Hence it defines an element w in the symmetric group S_d . The maps $\{1, \dots, d\} \rightarrow A_0 = \mathbb{Z}^{\oplus d}/\mathbb{Z}(1, \dots, 1)$ which sends i to the class of $x_{w^{-1}(i)}$ in A_0 gives an element $[\tilde{\sigma}]$ in $T(\sigma)$.

Lemma 4. *The element $[\tilde{\sigma}] \in T(\sigma)$ does not depend on the choice of a lift $\tilde{\sigma}$.*

Proof. For each integer $j \in \mathbb{Z}$ we write j of the form $j = md + r$ with $m \in \mathbb{Z}$ and $r \in \{0, \dots, d-1\}$ and set $x_j = x_r + m(1, \dots, 1)$. As we have mentioned in the proof of Lemma 3, The inverse image of σ under the canonical surjection $\mathbb{Z}^{\oplus d} \rightarrow \mathbb{Z}^{\oplus d}/\mathbb{Z}(1, \dots, 1)$ is equal to $\{x_j \mid j \in \mathbb{Z}\}$ and any small lift $\tilde{\sigma}'$ of σ to $\mathbb{Z}^{\oplus d}$ is of the form $\tilde{\sigma}' = \{x_l, x_{l+1}, \dots, x_{l+d-1}\}$ for some $l \in \mathbb{Z}$. For each $i \in \{1, \dots, d\}$, the unique integer $j \in \{l, l+1, \dots, l+d-1\}$ such that the i -th coordinate of $x_j - x_{j-1}$ is equal to 1 and the other coordinates of $x_j - x_{j-1}$ are equal to zero is congruent to $w^{-1}(i)$ modulo d . Hence the class of x_j in A_0 does not depend on the choice of a small lift $\tilde{\sigma}'$. This proves the claim. \square

We denote by $[\sigma]$ the class of $[\tilde{\sigma}]$ in $O(\sigma)$. We let $\tilde{\beta}$ denote the element $\tilde{\beta} = (\beta_{\nu})_{\nu \in A'_{d-1}}$ in $\prod_{\nu \in A'_{d-1}} \mathbb{Z}$ where $\beta_{\nu} = 1$ if $\nu = [\sigma]$ for some $\sigma \in A_{d-1}$ and $\beta_{\nu} = 0$ otherwise. We denote by β the class of $\tilde{\beta}$ in $(\prod_{\nu \in A'_{d-1}} \mathbb{Z})_{\{\pm 1\}}$.

Proposition 5. *The element $\beta \in (\prod_{\nu \in A'_{d-1}} \mathbb{Z})_{\{\pm 1\}}$ is a $(d-1)$ -cycle in the chain complex which computes the Borel-Moore homology of A_{\bullet} .*

Proof. The assertion is clear for $d = 1$ since the $(d-2)$ -nd component of the complex is zero. Suppose that $d \geq 2$. Let τ be an element in A_{d-2} . Take a small lift $\tilde{\tau} \subset \mathbb{Z}^{\oplus d}$ of τ to $\mathbb{Z}^{\oplus d}$. By definition the set $\tilde{\tau}$ is of the form $\tilde{\tau} = \{x_1, \dots, x_d\}$ with $x_0 \leq x_1 \leq \dots \leq x_{d-1}$ where we have set $x_0 = x_d - (1, \dots, 1)$. There is a unique $i \in \{1, \dots, d-1\}$ such that $x_i - x_{i-1}$ has two non-zero coordinates. There are exactly two elements in $\mathbb{Z}^{\oplus d}$ which is larger than x_{i-1} and which is smaller than x_i . We denote these two elements by y_1 and y_2 . We set $\tilde{\sigma}_j = \tilde{\tau} \cup \{y_j\}$ for $j = 1, 2$. The sets $\tilde{\sigma}_1, \tilde{\sigma}_2$ are small subsets of $\mathbb{Z}^{\oplus d}$ of cardinality d and their images σ_1, σ_2 under the surjection $\mathbb{Z}^{\oplus d} \rightarrow \mathbb{Z}^{\oplus d}/\mathbb{Z}(1, \dots, 1)$ are elements in A_{d-1} . For $j = 1, 2$, let w_j denote the element w in the symmetric group S_d which appeared in the first paragraph of Section 3.2.3 for $\sigma = \sigma_j$. It follows from the definition of σ_j that we have $w_1 = w_2(i, i+1)$, where $(i, i+1)$ denotes the transposition of i and $i+1$. It is easily checked that the set of the elements in A_{d-1} which has τ as a face is equal to $\{\sigma_1, \sigma_2\}$. Since we have $\text{sgn}(w_1) = -\text{sgn}(w_2)$, it follows that the component in $(\prod_{\nu \in O(\tau)} \mathbb{Z})_{\{\pm 1\}}$ of the image of β under the boundary map $(\prod_{\nu \in A'_{d-1}} \mathbb{Z})_{\{\pm 1\}} \rightarrow (\prod_{\nu' \in A'_{d-2}} \mathbb{Z})_{\{\pm 1\}}$ is equal to zero. This proves the claim. \square

4. ARITHMETIC GROUPS AND MODULAR SYMBOLS

4.1. Arithmetic groups.

4.1.1. *An arithmetic group.* We give here the definition of our main object of study, an arithmetic group Γ .

Let us give the setup. We let F denote a global field of positive characteristic. Let C be a proper smooth curve over a finite field whose function field is F . Let ∞ be a place of F and let $K = F_\infty$ denote the local field at ∞ . We let $A = H^0(C \setminus \{\infty\}, \mathcal{O}_C)$. Here we identified a closed point of C and a place of F . We write $\hat{A} = \varprojlim_I A/I$, where the limit is taken over the nonzero ideals of A . We let $\mathbb{A}^\infty = \hat{A} \otimes_A F$ denote the ring of finite adeles.

Let $\mathbb{K}^\infty \subset \mathrm{GL}_d(\mathbb{A}^\infty)$ be a compact open subgroup. We set $\Gamma = \mathrm{GL}_d(F) \cap \mathbb{K}^\infty$ and regard it as a subgroup of $\mathrm{GL}_d(K)$. We refer to the group of this form for some \mathbb{K}^∞ an arithmetic (sub)group of $\mathrm{GL}_d(K)$ (contained in $\mathrm{GL}_d(F)$).

We give a remark. Let Γ be an arithmetic group. Then $\Gamma \cap \mathrm{SL}_d(F) = \Gamma \cap \mathrm{SL}_d(K)$ is a subgroup of Γ of finite index, and is an S -arithmetic group of SL_d over F for $S = \{\infty\}$ dealt in the paper of Harder [Har].

4.1.2. Let $\Gamma \subset \mathrm{GL}_d(K)$ be a subgroup. We consider the following Conditions (1) to (5) on Γ .

- (1) $\Gamma \subset \mathrm{GL}_d(K)$ is a discrete subgroup,
- (2) $\{\det(\gamma) \mid \gamma \in \Gamma\} \subset O_\infty^\times$ where O_∞ is the ring of integers of K ,
- (3) $\Gamma \cap Z(\mathrm{GL}_d(K))$ is finite.

Let $A_\bullet = A_{v_1, \dots, v_d, \bullet}$ denote the apartment corresponding to a basis $v_1, \dots, v_d \in K^{\oplus d}$ (defined in Section 3.2.2).

- (4) For any apartment $A_\bullet = A_{v_1, \dots, v_d, \bullet}$ with $v_1, \dots, v_d \in F^{\oplus d}$, the composition $A_\bullet \hookrightarrow \mathcal{BT}_\bullet \rightarrow \Gamma \backslash \mathcal{BT}_\bullet$ is quasi-finite, that is, the inverse image of any simplex by this map is a finite set.
- (5) The cohomology group $H^{d-1}(\Gamma, \mathbb{Q})$ is a finite dimensional \mathbb{Q} -vector space.

The condition (1) will be used in Lemma 13. The condition (2) implies that each element in the isotropy group of a simplex fixes the vertices of the simplex. Under the condition (1), the condition (3) implies that the stabilizer of a simplex is finite. This implies that the \mathbb{Q} -coefficient group homology of Γ and the homology of $\Gamma \backslash |\mathcal{BT}_\bullet|$ are isomorphic. The condition (4) will be used to define a class in Borel-Moore homology of $\Gamma \backslash \mathcal{BT}_\bullet$ starting from an apartment (Section 4.3). The condition (5) will be used in the proof of Lemma 16.

Let us show that all five conditions of Section 4.1.2 are satisfied when Γ is an arithmetic subgroup. The condition (1) holds trivially. We note that there exists an element $g \in \mathrm{GL}_d(\mathbb{A}^\infty)$ such that $g\mathbb{K}^\infty g^{-1} \subset \mathrm{GL}_d(\hat{A})$. Since $\det(\gamma) \in F^\times \cap \hat{A}^\times \subset O_\infty^\times$ for $\gamma \in \Gamma$, (2) holds. Because $F^\times \cap \mathrm{GL}_d(\hat{A})$ is finite, (3) holds.

Lemma 6. *Let Γ be an arithmetic subgroup. Then (4) holds.*

Proof. We show that the inverse image of each simplex of $\Gamma \backslash \mathcal{BT}_\bullet$ under the map in (4) is finite. The set of simplices of \mathcal{BT}_\bullet of a fixed dimension is identified (see Section 6.1.2 for the identification) with the coset $\mathrm{GL}_d(K)/\tilde{\mathbb{K}}_\infty$ for an open subgroup $\tilde{\mathbb{K}}_\infty \subset \mathrm{GL}_d(K)$ which contains $K^\times \mathbb{K}_\infty$ as a subgroup of finite index for some compact open subgroup $\mathbb{K}_\infty \subset \mathrm{GL}_d(K)$. Let $T \subset \mathrm{GL}_d$ denote the diagonal maximal torus.

The set of simplices of A_\bullet of fixed dimension is identified with the image of the map

$$\coprod_{w \in S_d} gwT(K) \rightarrow \mathrm{GL}_d(K)/\tilde{\mathbb{K}}_\infty$$

for some $g \in \mathrm{GL}_d(F)$. Since S_d is a finite group, it then suffices to show that for any $w \in S_d$, the map

$$\mathrm{Image}[gwT(K) \rightarrow \mathrm{GL}_d(K)/K^\times \mathbb{K}_\infty] \rightarrow \Gamma \backslash \mathrm{GL}_d(K)/K^\times \mathbb{K}_\infty$$

is quasi-finite. The inverse image under the last map of the image of $gwt \in gwT(K)$ is isomorphic to the set

$$\begin{aligned} \{\gamma \in \Gamma \mid gwt \in gwT(K)K^\times \mathbb{K}_\infty\} &= \Gamma \cap gwT(K)K^\times \mathbb{K}_\infty(gwt)^{-1} \\ &= \Gamma \cap (gw)T(K)t\mathbb{K}_\infty t^{-1}(gw)^{-1}. \end{aligned}$$

Hence, if we let $g' = gw$ and $\mathbb{K}'_\infty = t\mathbb{K}_\infty t^{-1}$, this set equals

$$\begin{aligned} \Gamma \cap g'T(K)\mathbb{K}'_\infty g'^{-1} &= \mathrm{GL}_d(F) \cap (\mathbb{K}^\infty \times g'T(K)\mathbb{K}'_\infty g'^{-1}) \\ &= g'(\mathrm{GL}_d(F) \cap (g'^{-1}\mathbb{K}^\infty g' \cap T(K)\mathbb{K}'_\infty)g'^{-1}). \end{aligned}$$

The finiteness is proved in the following lemma. □

Lemma 7. *For any compact open subgroup $\mathbb{K} \subset \mathrm{GL}_d(\mathbb{A})$, the set $\mathrm{GL}_d(F) \cap T(K)\mathbb{K}$ is finite.*

Proof. Let $U = T(K) \cap \mathbb{K}$. Then $T(O_\infty) \supset U$ and is of finite index. Note that there exist a non-zero ideal $I \subset A$ and an integer N such that $\mathbb{K} \subset I^{-1}\varpi_\infty^{-N}\mathrm{Mat}_d(\hat{A}) \times \mathrm{Mat}_d(O_\infty)$ where ϖ_∞ is a uniformizer in O_∞ .

Let $\alpha : T(K)/U \rightarrow T(K)/T(O_\infty) \cong \mathbb{Z}^{\oplus d}$ be the (quasi-finite) map induced by the inclusion $U \subset T(O_\infty)$. For $h \in T(K)$, we write $(h_1, \dots, h_d) = \alpha(h)$. Then for $i = 1, \dots, d$, the i -th row of $h\mathbb{K}$ is contained in

$(I^{-1}\widehat{A} \times \varpi_{\infty}^{-N}\varpi_{\infty}^{h_i}O_{\infty})^{\oplus d}$. Hence, for sufficiently large h_i , the intersection $h\mathbb{K} \cap \mathrm{GL}_d(F)$ is empty. We then have, for sufficiently large N' ,

$$(4.1) \quad \mathrm{GL}_d(F) \cap T(K)\mathbb{K} = \coprod_{\substack{h \in T(K)/U, \\ h_1, \dots, h_d \leq N'}} \mathrm{GL}_d(F) \cap h\mathbb{K}.$$

The adelic norm of the determinant of an element in $\mathrm{GL}_d(F)$ is 1, while that of an element in $h\mathbb{K}$ is $|\det h|_{\infty} = \sum_{i=1}^d h_i$. So (4.1) equals

$$\coprod_{h \in T(K)/U, h_i \leq N', \sum h_i = 0} \mathrm{GL}_d(F) \cap h\mathbb{K}.$$

The index set of the disjoint union above is finite since α is quasi-finite, and $\mathrm{GL}_d(F) \cap h\mathbb{K}$ is finite since $\mathrm{GL}_d(F)$ is discrete and $h\mathbb{K}$ is compact. The claim follows. \square

Lemma 8. *Let Γ be an arithmetic subgroup. Then (5) holds.*

Proof. This follows from [Har, p.136, Satz 2]. \square

4.2. Arithmetic quotient of the Bruhat-Tits building. Let us define simplicial complex $\Gamma \backslash \mathcal{BT}_{\bullet}$ for an arithmetic subgroup Γ in this section.

We need a lemma.

Lemma 9. *Let $i \geq 0$ be an integer, let $\sigma \in \mathcal{BT}_i$ and let $v, v' \in V(\sigma)$ be two vertices with $v \neq v'$. Suppose that an element $g \in \mathrm{GL}_d(K)$ satisfies $|\det g|_{\infty} = 1$. Then we have $gv \neq v'$.*

Proof. Let $\tilde{\sigma}$ be an element $(L_j)_{j \in \mathbb{Z}}$ in $\widetilde{\mathcal{BT}}_i$ such that the class of $\tilde{\sigma}$ in \mathcal{BT}_i is equal to σ . There exist two integers $j, j' \in \mathbb{Z}$ such that v, v' is the class of $L_j, L_{j'}$, respectively. Assume that $gv = v'$. Then there exists an integer $k \in \mathbb{Z}$ such that $L_j g^{-1} = \varpi_{\infty}^k L_{j'} = L_{j' + (i+1)k}$. Let us fix a Haar measure $d\mu$ of the K -vector space $V_{\infty} = K^{\oplus d}$. As is well-known, the push-forward of $d\mu$ with respect to the automorphism $V_{\infty} \rightarrow V_{\infty}$ given by the right multiplication by γ is equal to $|\det \gamma|_{\infty}^{-1} d\mu$ for every $\gamma \in \mathrm{GL}_d(K)$. Since $|\det g|_{\infty} = 1$, it follows from the equality $L_j g^{-1} = L_{j' + (i+1)k}$ that the two \mathcal{O}_{∞} -lattices L_j and $L_{j' + (i+1)k}$ have a same volume with respect to $d\mu$. Hence we have $j = j' + (i+1)k$, which implies $L_j = \varpi_{\infty}^k L_{j'}$. It follows that the class of L_j in \mathcal{BT}_0 is equal to the class of $L_{j'}$, which contradicts the assumption $v \neq v'$. \square

Let $\Gamma \subset \mathrm{GL}_d(K)$ be an arithmetic subgroup.

It follows from Lemma 9 (using Condition (2) of Section 4.1.2) that for each $i \geq 0$ and for each $\sigma \in \mathcal{BT}_i$, the image of $V(\sigma)$ under the surjection $\mathcal{BT}_0 \rightarrow \Gamma \backslash \mathcal{BT}_0$ is a subset of $\Gamma \backslash \mathcal{BT}_0$ with cardinality $i+1$. We denote this subset by $V(\mathrm{cl}(\sigma))$, since it is easily checked that it depends only on the class $\mathrm{cl}(\sigma)$ of σ in $\Gamma \backslash \mathcal{BT}_i$. Thus the collection $\Gamma \backslash \mathcal{BT}_{\bullet} = (\Gamma \backslash \mathcal{BT}_i)_{i \geq 0}$ has a canonical structure of a simplicial complex such that the collection of the canonical surjection $\mathcal{BT}_i \rightarrow \Gamma \backslash \mathcal{BT}_i$ is a map of simplicial complexes $\mathcal{BT}_{\bullet} \rightarrow \Gamma \backslash \mathcal{BT}_{\bullet}$.

4.3. Modular symbols. Let v_1, \dots, v_d be an F -basis (that is, a basis of $F^{\oplus d}$ regarded as a basis of $K^{\oplus d}$). We consider the composite

$$(4.2) \quad A_{\bullet} \xrightarrow{\iota_{v_1, \dots, v_d}} \mathcal{BT}_{\bullet} \rightarrow \Gamma \backslash \mathcal{BT}_{\bullet}.$$

Condition (4) implies that the map (4.2) is a finite map of simplicial complexes in the sense of Section 2.2.4. It follows that the map (4.2) induces a homomorphism

$$H_{d-1}^{\mathrm{BM}}(A_{\bullet}, \mathbb{Z}) \rightarrow H_{d-1}^{\mathrm{BM}}(\Gamma \backslash \mathcal{BT}_{\bullet}, \mathbb{Z}).$$

We let $\beta_{v_1, \dots, v_d} \in H_{d-1}^{\mathrm{BM}}(\Gamma \backslash \mathcal{BT}_{\bullet}, \mathbb{Z})$ denote the image under this homomorphism of the element $\beta \in H_{d-1}^{\mathrm{BM}}(A_{\bullet}, \mathbb{Z})$ introduced in Section 3.2.3. We call this the class of the apartment $A_{v_1, \dots, v_d, \bullet}$.

4.4. Main Theorem. We are ready to state our theorem.

Theorem 10. *Let $\Gamma \subset \mathrm{GL}_d(K)$ be an arithmetic subgroup. The image of the canonical map (see Section 2.2.2 for the definition)*

$$H_{d-1}(\Gamma \backslash \mathcal{BT}_{\bullet}, \mathbb{Q}) \rightarrow H_{d-1}^{\mathrm{BM}}(\Gamma \backslash \mathcal{BT}_{\bullet}, \mathbb{Q})$$

is contained in the sub \mathbb{Q} -vector space generated by the classes of apartments associated to F -bases.

5. PROOF OF THEOREM 10

The purpose of this section is to prove Theorem 10. It gives the description of the homology of certain arithmetic groups in terms of the subspace generated by the classes of modular symbols inside the Borel-Moore homology of the quotient of the Bruhat-Tits building. The treatment of the modular symbols differs from the archimedean case (see [As-Ru]) in that the group does not act freely on the compactification and in that an apartment is contractible as a subspace of the building. To compare, we use equivariant homology (Section 5.1) of Werner's compactification (Section 5.2) as an intermediary object.

5.1. Equivariant homology. Let $\Gamma \subset \mathrm{GL}_d(K)$ be an arithmetic subgroup. We define the simplicial set (not a simplicial complex) ET_\bullet as follows. We define $ET_n = \Gamma^{n+1}$ to be the $(n+1)$ -fold direct product of Γ for $n \geq 0$. The set Γ^{n+1} is naturally regarded as the set of maps of sets $\mathrm{Map}(\{0, \dots, n\}, \Gamma)$ and from this one obtains naturally the structure of a simplicial set. We let $|ET_\bullet|$ denote the geometric realization of ET_\bullet . Then $|ET_\bullet|$ is contractible. We let Γ act diagonally on each ET_n ($n \geq 0$). The induced action on $|ET_\bullet|$ is free.

Let M be a topological space on which Γ acts. The diagonal action of Γ on $M \times |ET_\bullet|$ is free. We let $H_*^\Gamma(M, B) = H_*(\Gamma \backslash (M \times |ET_\bullet|), B)$ where B is a coefficient ring, and call it the equivariant homology of M with coefficients in B . We also use the relative version, and define equivariant cohomology in a similar manner.

5.2. Werner's compactification. In this section, we briefly recall the result of Werner ([We1], [We2]).

5.2.1. Semi-norms. Let W be an K -vector space. We call a function $\gamma : W \rightarrow \mathbb{R}_{\geq 0}$ a semi-norm if the following conditions are satisfied:

- (1) $\gamma(\lambda w) = |\lambda| \gamma(w)$ for $\lambda \in F_\infty, w \in W$,
- (2) $\gamma(w_1 + w_2) \leq \sup\{\gamma(w_1), \gamma(w_2)\}$ for $w_1, w_2 \in W$,
- (3) There exists an element $w \in W$ satisfying $\gamma(w) \neq 0$.

We say that two semi-norms are equivalent if and only if one is a non-zero constant multiple of the other.

Let $V^* = \mathrm{Hom}_K(V, K)$ be the dual vector space of V . We endow the set S' of semi-norms on V^* with the topology of pointwise convergence. We give the set S of equivalence classes of semi-norms the quotient topology.

5.2.2. We write $\overline{|\mathcal{BT}_\bullet|}$ for the compactification of $|\mathcal{BT}_\bullet|$ of Werner in [We1] (which uses lattices of smaller rank), and let $\partial \overline{|\mathcal{BT}_\bullet|} = \overline{|\mathcal{BT}_\bullet|} \setminus |\mathcal{BT}_\bullet|$. The topological space $\overline{|\mathcal{BT}_\bullet|}$ is compact and contractible ([We1, p.519, Theorem 4.1]). The action of $\mathrm{GL}(V)$ is extended to $\overline{|\mathcal{BT}_\bullet|}$ ([We2, Theorem 4.2]).

By a theorem of Goldman-Iwahori (see [De-Hu, Theorem 2.2]), the set of equivalence classes of norms on V^* is isomorphic to the set of points of the geometric realization of the Bruhat-Tits building for $\mathrm{PGL}(V^*)$. In the paper of Werner [We2, Theorem 5.1], this isomorphism is extended to a canonical homeomorphism $S \cong \overline{|\mathcal{BT}_{V^*, \bullet}|}$ where $\overline{|\mathcal{BT}_{V^*, \bullet}|}$ is the compactification of $|\mathcal{BT}_{V^*, \bullet}|$ using semi-norms. We use the homeomorphism $\overline{|\mathcal{BT}_{V^*, \bullet}|} \cong \overline{|\mathcal{BT}_{V, \bullet}|}$ of Werner ([We2] p.518), and obtain a homeomorphism $S \cong \overline{|\mathcal{BT}_{V, \bullet}|}$.

5.3. Let us give an outline of the proof of Theorem 10 in this section. We construct the following commutative diagram in Sections 5.4 and 5.5:

$$\begin{array}{ccc}
 H_{d-1}(\Gamma \backslash \mathcal{BT}_\bullet, \mathbb{Q}) & \xrightarrow{(1)} & H_{d-1}^{\mathrm{BM}}(\Gamma \backslash \mathcal{BT}_\bullet, \mathbb{Q}) \\
 (2) \uparrow \cong & & \uparrow (3) \\
 H_{d-1}^\Gamma(\overline{|\mathcal{BT}_\bullet|}, \mathbb{Q}) & \xrightarrow{(5)} & H_{d-1}^\Gamma(\overline{|\mathcal{BT}_\bullet|}, \partial \overline{|\mathcal{BT}_\bullet|}; \mathbb{Q}) \\
 (4) \uparrow \cong & & \\
 H_{d-1}(\Gamma, \mathbb{Q}) & &
 \end{array}$$

Here the map (1) is the map that appeared in the statement of Theorem 10. The map (5) is the pushforward map of homology. The other maps will be constructed later. It is easy to see that the groups $H_{d-1}(\Gamma, \mathbb{Q})$ and $H_{d-1}(\Gamma \backslash \overline{|\mathcal{BT}_\bullet|} \times |ET_\bullet|, \mathbb{Q})$ are isomorphic since $\overline{|\mathcal{BT}_\bullet|} \times |ET_\bullet|$ is contractible and Γ acts freely. However, the key here is to construct (4) explicitly at the level of chain complexes in the direction indicated by the arrow above, so that we are able to compute explicitly the image of the composite $(5) \circ (4)$.

The construction of the square is elementary, but there is one problem which is caused by that the isomorphism between the Borel-Moore homology as defined in this paper and the Borel-Moore homology of a topological space in general is not found in the literature. We resort to the well-known cases of the

isomorphisms for homology and for cohomology to circumvent this problem. In order to do so, we use property (5) of the arithmetic group (Section 4.1.2) and take the dual twice.

5.4. On the maps (4) and (5). We construct the map (4) and show that it is an isomorphism. This is done very explicitly, so that we are able to compute the image of the composite map (5)(4) (Lemma 13).

5.4.1. We let C_\bullet denote the complex of $\mathbb{Z}[\Gamma]$ -modules defined by $C_n = \mathbb{Z}[\Gamma^{n+1}]$ ($n \geq 0$) and the usual boundary homomorphisms. It is a free resolution of the trivial $\mathbb{Z}[\Gamma]$ -module \mathbb{Z} . The homology group of the Γ -coinvariants $C_{\Gamma,\bullet}$ of C_\bullet is the group homology $H_*(\Gamma, \mathbb{Z})$.

Let $D_n = \mathbb{Z}[\text{Map}_{\text{cont}}(\Delta_n, \overline{|\mathcal{BT}_\bullet|} \times |E\Gamma_\bullet|)]$. The usual boundary map turns D_\bullet into a complex of $\mathbb{Z}[\Gamma]$ -modules with Γ acting on $\overline{|\mathcal{BT}_\bullet|} \times |E\Gamma_\bullet|$ diagonally. It is a free resolution of the trivial $\mathbb{Z}[\Gamma]$ -module since the action of Γ on $\overline{|\mathcal{BT}_\bullet|} \times |E\Gamma_\bullet|$ is free.

The Γ -coinvariants, denoted $D_{\Gamma,\bullet}$, is canonically isomorphic to the module

$$\mathbb{Z}[\text{Map}_{\text{cont}}(\Delta_n, \Gamma \backslash \overline{|\mathcal{BT}_\bullet|} \times |E\Gamma_\bullet|)].$$

Hence the homology group of the complex $D_{\Gamma,\bullet}$ is $H_*^\Gamma(\overline{|\mathcal{BT}_\bullet|}, \mathbb{Z})$.

Let $r \geq 0$ be an integer. Let $\Delta_r = \{(t_0, \dots, t_r) \in \mathbb{R}^{r+1} \mid \sum t_i = 1, 0 \leq t_i \leq 1\}$ be the (geometric) r -simplex. Given $v_0, \dots, v_r \in V \setminus \{0\}$, we construct a map $s(v_0, \dots, v_r) : \Delta_r \rightarrow S \cong \overline{|\mathcal{BT}_\bullet|}$ as follows. For $(t_0, \dots, t_r) \in \Delta_r$ and $f \in V^*$, we set

$$s(v_0, \dots, v_r)(t_0, \dots, t_r)(f) = \sup_{0 \leq i \leq r} |f(v_i)| q_\infty^{-1/t_i}.$$

Here, we set $q_\infty^{-1/t_i} = 0$ if $t_i = 0$. It is easy to check that $s(v_0, \dots, v_r)(t_0, \dots, t_r)$ is a semi-norm on V^* for each $(t_0, \dots, t_r) \in \Delta_r$.

Lemma 11. *The map $s(v_0, \dots, v_r)$ is continuous.*

Proof. This is immediate from the definition of the topology on S' and on S , since for each $f \in V^*$, the map $s(v_0, \dots, v_r)(t_0, \dots, t_r)(f) : \Delta_r \rightarrow \mathbb{R}_{\geq 0}$ is continuous. \square

5.4.2. Let $r \geq 0$ be an integer. Given $v \in V \setminus \{0\}$ and $[g_0, \dots, g_r] \in \mathbb{Z}[\Gamma^{r+1}] = C_r$, we set $\eta_v([g_0, \dots, g_r]) = s(g_0 v, \dots, g_r v) \times [g_0, \dots, g_r] : \Delta_r \rightarrow \overline{|\mathcal{BT}_\bullet|} \times |E\Gamma_\bullet|$. Here $[g_0, \dots, g_r]$ on the right hand side is regarded as the canonical inclusion $\Delta_r \hookrightarrow |E\Gamma_\bullet|$ associated to the r -simplex (g_0, \dots, g_r) of $E\Gamma_\bullet$. Extending this by linearity, we obtain a map of complexes $\eta_v : C_\bullet \rightarrow D_\bullet$.

Lemma 12. *The map η_v is a quasi-isomorphism.*

Proof. We have seen that both C_\bullet and D_\bullet are free $\mathbb{Z}[\Gamma]$ -resolutions of the trivial $\mathbb{Z}[\Gamma]$ -module \mathbb{Z} . So we only need to check at degree 0, that is, the commutativity of the following diagram:

$$\begin{array}{ccc} C_0 & \xrightarrow{\eta_v} & D_0 \\ \downarrow & & \downarrow \\ \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \end{array}$$

where the vertical homomorphisms are augmentations. This is clear. \square

Taking Γ -coinvariants, we obtain a map of complexes $C_{\Gamma,\bullet} \rightarrow D_{\Gamma,\bullet}$. It induces a map of homology $H_*(\Gamma, \mathbb{Z}) \rightarrow H_*^\Gamma(\overline{|\mathcal{BT}_\bullet|}, \mathbb{Z})$. This is an isomorphism since both C_\bullet and D_\bullet are free $\mathbb{Z}[\Gamma]$ -resolutions of \mathbb{Z} . We define the map (4) to be this map tensored by \mathbb{Q} .

5.4.3. For $n \geq 0$, let

$$\tilde{D}_{\Gamma,n} = \mathbb{Z}[\text{Map}_{\text{cont}}(\Delta_n, \Gamma \backslash \overline{|\mathcal{BT}_\bullet|} \times |E\Gamma_\bullet|)] / \mathbb{Z}[\text{Map}_{\text{cont}}(\Delta_n, \Gamma \backslash \partial \overline{|\mathcal{BT}_\bullet|} \times |E\Gamma_\bullet|)].$$

Then $H_*^\Gamma(\overline{|\mathcal{BT}_\bullet|}, \partial \overline{|\mathcal{BT}_\bullet|}; \mathbb{Z})$ is the homology group of the complex $\tilde{D}_{\Gamma,\bullet}$. The canonical surjection at each degree induces a map of complexes $D_{\Gamma,\bullet} \rightarrow \tilde{D}_{\Gamma,\bullet}$, and a homomorphism $H_*^\Gamma(\overline{|\mathcal{BT}_\bullet|}, \mathbb{Z}) \rightarrow H_*^\Gamma(\overline{|\mathcal{BT}_\bullet|}, \partial \overline{|\mathcal{BT}_\bullet|}; \mathbb{Z})$. The map (5) of the diagram (5.1) is this map tensored by \mathbb{Q} .

Let $v_1, \dots, v_d \in V$ be a basis and let $g_0, \dots, g_{d-1} \in \Gamma$. By construction, the image of the faces of Δ_{d-1} by the continuous map

$$s(v_1, \dots, v_d) \times [g_0, \dots, g_{d-1}] : \Delta_{d-1} \rightarrow \overline{|\mathcal{BT}_\bullet|} \times |E\Gamma_\bullet|$$

is contained in $\partial \overline{|\mathcal{BT}_\bullet|} \times |E\Gamma_\bullet|$. We let $A_{v_1, \dots, v_d; g_0, \dots, g_{d-1}}$ denote the class of this continuous function in $\tilde{D}_{\Gamma,d-1}$ and in $H_{d-1}^\Gamma(\overline{|\mathcal{BT}_\bullet|}, \partial \overline{|\mathcal{BT}_\bullet|}; \mathbb{Z})$. We let $\mathcal{A}_F^{\text{rel}}$ denote the submodule of $H_{d-1}^\Gamma(\overline{|\mathcal{BT}_\bullet|}, \partial \overline{|\mathcal{BT}_\bullet|}; \mathbb{Z})$ generated by elements of the form $A_{v_1, \dots, v_d; g_0, \dots, g_{d-1}}$ with $g_i \in \Gamma$ ($0 \leq i \leq d-1$) and $v_1, \dots, v_d \in V_F = F^{\oplus d} \subset K^{\oplus d} = V$ an F -basis.

Lemma 13. *The image of*

$$H_{d-1}(\Gamma, \mathbb{Q}) \xrightarrow{(4)} H_{d-1}(\overline{|\mathcal{BT}_\bullet|}, \mathbb{Q}) \xrightarrow{(5)} H_{d-1}(\overline{|\mathcal{BT}_\bullet|}, \partial\overline{|\mathcal{BT}_\bullet|}; \mathbb{Q})$$

is contained in the sub \mathbb{Q} -vector space generated by $\mathcal{A}_F^{\text{rel}}$.

Proof. Take a $v \in V_F \setminus \{0\} \subset V \setminus \{0\}$. Consider the map of complexes $C_{\Gamma, \bullet} \rightarrow D_{\Gamma, \bullet} \rightarrow \widetilde{D}_{\Gamma, \bullet}$ where the first map is η_v and the second map is the canonical map. The image of $C_{\Gamma, d-1}$ is of the form

$$s(g_0 v, \dots, g_{d-1} v) \times [g_0, \dots, g_{d-1}]$$

for some $g_0, \dots, g_{d-1} \in \Gamma$. Since $v \in F^{\oplus d}$ and $g_0, \dots, g_{d-1} \in \Gamma \subset \text{GL}_d(F)$ by the condition (1) on Γ , the vectors $g_0 v, \dots, g_{d-1} v$ are F -vectors. If $g_0 v, \dots, g_{d-1} v$ do not form a basis, then the element above is zero in $H_{d-1}^\Gamma(\overline{|\mathcal{BT}_\bullet|}, \partial\overline{|\mathcal{BT}_\bullet|}; \mathbb{Q})$ because by the construction of s the image of the map above is contained in $\Gamma \setminus \partial\overline{|\mathcal{BT}_\bullet|} \times |\mathcal{ET}_\bullet|$. \square

5.5. The maps (2) and (3). Given a \mathbb{Q} -vector space A , let $A^* = \text{Hom}(A, \mathbb{Q})$ denote the dual. In what follows, the coefficient ring is \mathbb{Q} unless otherwise specified.

5.5.1. Consider the following diagram.

$$(5.2) \quad \begin{array}{ccc} H_{d-1}(\Gamma \setminus \mathcal{BT}_\bullet) & \xrightarrow{(1)} & H_{d-1}^{\text{BM}}(\Gamma \setminus \mathcal{BT}_\bullet) \\ \cong \downarrow (6) & & \downarrow (7) \\ H_{d-1}(\Gamma \setminus \mathcal{BT}_\bullet)^{**} & & \cong \\ \downarrow = (8) & & \downarrow \\ H^{d-1}(\Gamma \setminus \mathcal{BT}_\bullet)^* & \xrightarrow{(9)} & H_c^{d-1}(\Gamma \setminus \mathcal{BT}_\bullet)^* \end{array}$$

The map (1) is the canonical map from homology to Borel-Moore homology. The map (6) is the canonical map $A \rightarrow A^{**}$ for $A = H_{d-1}(\Gamma \setminus \mathcal{BT}_\bullet)$. We will see later (Corollary 17) that (6) is an isomorphism. The map (7) is the isomorphism given by the map in the universal coefficient theorem (see Section 2.2.3). The map (8) is the dual of the fact that cohomology is the dual of homology. The map (9) is the dual of the canonical map from cohomology with compact support to cohomology. It follows from the definitions that the diagram is commutative.

5.5.2. Consider the diagram

$$(5.3) \quad \begin{array}{ccc} H^{d-1}(\Gamma \setminus \mathcal{BT}_\bullet) & \xleftarrow{(9)'} & H_c^{d-1}(\Gamma \setminus \mathcal{BT}_\bullet) \\ \cong \uparrow (10) & & \cong \downarrow (11) \\ & & \varinjlim_L H^{d-1}(\Gamma \setminus \mathcal{BT}_\bullet, L) \\ & & \cong \uparrow (12) \\ H^{d-1}(\Gamma \setminus |\mathcal{BT}_\bullet|) & \xleftarrow{(13)} \varinjlim_L H^{d-1}(\Gamma \setminus |\mathcal{BT}_\bullet|, |L|), \end{array}$$

where L runs over the subsimplicial complexes of $\Gamma \setminus \mathcal{BT}_\bullet$ such that the complement $|\Gamma \setminus \mathcal{BT}_\bullet| \setminus |L|$ is covered by a finite number of simplices.

The map (9)' is the forget support map, whose dual is the map (9) in diagram (5.2). The map (10) is the canonical map from singular cohomology to cellular cohomology (see Section 2.2.6). The map (11) is obtained from the definition. The map (12) at each stage is the canonical map from singular cohomology to cellular cohomology. The map (13) is the limit of the pullback maps. It is easy to check that the diagram is commutative.

Lemma 14. *The map (11) is an isomorphism.*

Proof. It suffices to show that, given a finite set B of simplices of $\Gamma \setminus \mathcal{BT}_\bullet$, there exists a subsimplicial complex $L \subset \Gamma \setminus \mathcal{BT}_\bullet$ such that

- (1) the cardinality of the set of simplices not contained in L is finite, that is, the cardinality of $((\Gamma \setminus \mathcal{BT}_\bullet) \setminus L) = \cup_{i \geq 0} ((\Gamma \setminus \mathcal{BT}_\bullet)_i \setminus L_i)$ is finite, and
- (2) $B \subset ((\Gamma \setminus \mathcal{BT}_\bullet) \setminus L)$.

Let us construct such an L . Let \overline{B} denote the set of simplices σ of $\Gamma \setminus \mathcal{BT}_\bullet$ such that

- there exists a simplex $\tau \in B$ such that σ and τ has a face in common.

Now we set L to be the set of simplices $\sigma \in \Gamma \backslash \mathcal{BT}_\bullet$ such that

- $\sigma \notin \overline{B}$, or
- there exists a simplex $\tau \in ((\Gamma \backslash \mathcal{BT}_\bullet) \setminus \overline{B})$ such that σ is a face of τ .

Then L has a structure of a subsimplicial complex. It is easy to see that

$$B \subset (\Gamma \backslash \mathcal{BT}_\bullet \setminus L) \subset \overline{B},$$

which implies (2) above. Since $\Gamma \backslash \mathcal{BT}_\bullet$ is locally finite, \overline{B} is a finite set, which implies (1). \square

5.5.3. Consider the following diagram:

$$(5.4) \quad \begin{array}{ccc} H^{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet|) & \xleftarrow{\beta} \varinjlim_L H^{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet|, |L|) & \\ \cong \downarrow \alpha & & \downarrow \alpha \\ H_\Gamma^{d-1}(|\mathcal{BT}_\bullet|) & \xleftarrow{\beta} \varinjlim_L H^{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet| \times |E\Gamma_\bullet|, |\widetilde{L}|) & \\ \cong \uparrow \alpha & & \cong \uparrow \alpha \\ H_\Gamma^{d-1}(|\overline{\mathcal{BT}_\bullet}|) & \xleftarrow{\beta} \varinjlim_L H^{d-1}(\Gamma \backslash |\overline{\mathcal{BT}_\bullet}| \times |E\Gamma_\bullet|, |\widetilde{L}| \cup \Gamma \backslash \partial |\overline{\mathcal{BT}_\bullet}| \times |E\Gamma_\bullet|) & \\ & \searrow \beta & \downarrow \beta \\ & & H^{d-1}(\Gamma \backslash |\overline{\mathcal{BT}_\bullet}| \times |E\Gamma_\bullet|, \Gamma \backslash \partial |\overline{\mathcal{BT}_\bullet}| \times |E\Gamma_\bullet|). \end{array}$$

Here L runs over the subsimplicial complexes as in diagram (5.3), and $|\widetilde{L}|$ is the inverse image of $|L|$ by the projection $\Gamma \backslash |\mathcal{BT}_\bullet| \times |E\Gamma_\bullet| \rightarrow \Gamma \backslash |\mathcal{BT}_\bullet|$. The maps labeled by α are (induced by) pullbacks. The maps β are (induced by) the forget support maps. The diagram is commutative.

The second vertical arrow on the right hand column is an isomorphism by the excision property of cohomology.

Lemma 15. *The maps on the left column of the diagram (5.4) are isomorphisms.*

Proof. As $|\mathcal{BT}_\bullet|$ is contractible and Γ satisfies (3) of Section 4.1.2 (hence the stabilizer group of a simplex is a finite group as discussed there), the group $H^{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet|)$ is isomorphic to $H^{d-1}(\Gamma)$. As $|\mathcal{BT}_\bullet| \times |E\Gamma_\bullet|$ and $|\overline{\mathcal{BT}_\bullet}| \times |E\Gamma_\bullet|$ are contractible and Γ acts freely, the groups $H^{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet| \times |E\Gamma_\bullet|)$ and $H^{d-1}(\Gamma \backslash |\overline{\mathcal{BT}_\bullet}| \times |E\Gamma_\bullet|)$ are also isomorphic to $H^{d-1}(\Gamma)$. (These statements can be proved using spectral sequences, which are compatible with pullbacks.) This implies that the left vertical arrows are isomorphisms. \square

5.5.4. Consider the following diagram.

$$(5.5) \quad \begin{array}{ccc} H^{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet| \times |E\Gamma_\bullet|)^* & \longrightarrow & H^{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet| \times |E\Gamma_\bullet|, \Gamma \backslash \partial |\overline{\mathcal{BT}_\bullet}| \times |E\Gamma_\bullet|)^* \\ \cong \uparrow & & \uparrow \\ H_{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet| \times |E\Gamma_\bullet|) & \longrightarrow & H_{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet| \times |E\Gamma_\bullet|, \Gamma \backslash \partial |\overline{\mathcal{BT}_\bullet}| \times |E\Gamma_\bullet|) \end{array}$$

Each of the two vertical arrows in the square is the canonical map of the form $A \rightarrow A^{**}$. The lower horizontal arrow is the pushforward map of homology. The top horizontal arrow is the twice dual of the lower horizontal arrow, and is the dual of the forget support map of cohomology.

Lemma 16. *The left vertical map in the diagram (5.5) is an isomorphism.*

Proof. As was remarked in the proof of Lemma 15, the group $H^{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet| \times |E\Gamma_\bullet|)$ is isomorphic to $H^{d-1}(\Gamma)$. From property (5) in Section 4.1.2, we know that $H^{d-1}(\Gamma)$ is a finite dimensional \mathbb{Q} -vector space. This implies the claim. \square

Corollary 17. *The map (6) in diagram (5.2) is an isomorphism.*

Proof. As remarked in the proof of the previous lemma, $H^{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet| \times |E\Gamma_\bullet|)$ is finite dimensional. Using the isomorphisms (8) (10) and the isomorphisms in diagrams (5.4) and (5.4), we see that $H_{d-1}(\Gamma \backslash \mathcal{BT}_\bullet)^{**}$ is also finite dimensional. The claim follows from this. \square

5.5.5. We define the map (2) in diagram (5.1) to be the composite

$$\begin{aligned} H_{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet| \times |E\Gamma_\bullet|) &\xrightarrow{\cong} H^{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet| \times |E\Gamma_\bullet|)^* \xleftarrow{\cong} H^{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet|)^* \\ &\xleftarrow{\cong} H^{d-1}(\Gamma \backslash \mathcal{BT}_\bullet)^* \xleftarrow{\cong} H_{d-1}(\Gamma \backslash \mathcal{BT}_\bullet) \end{aligned}$$

where the maps are the left vertical arrow in the diagram (5.5), the dual of the composite of the left vertical arrows in (5.4), the dual of (10), and the dual of the composite (8)(6).

We define the map (3) in diagram (5.1) to be the composite

$$\begin{aligned} H_{d-1}(\Gamma \backslash |\overline{\mathcal{BT}_\bullet}| \times |E\Gamma_\bullet|, \Gamma \backslash \partial |\overline{\mathcal{BT}_\bullet}| \times |E\Gamma_\bullet|) &\rightarrow H^{d-1}(\Gamma \backslash |\overline{\mathcal{BT}_\bullet}| \times |E\Gamma_\bullet|, \Gamma \backslash \partial |\overline{\mathcal{BT}_\bullet}| \times |E\Gamma_\bullet|)^* \\ &\rightarrow \varinjlim_L H^{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet|, |L|) \rightarrow H_c^{d-1}(\Gamma \backslash \mathcal{BT}_\bullet)^* \rightarrow H_{d-1}^{\text{BM}}(\Gamma \backslash \mathcal{BT}_\bullet) \end{aligned}$$

where the maps are the right vertical arrow in the diagram (5.5), the dual of the vertical arrows in the diagram (5.4), the dual of the composite (11)⁻¹(12) and the inverse of (7).

The diagram (5.1) is then commutative since each of the diagrams (5.2), (5.3), (5.4), (5.5) is commutative.

5.6. Given a basis v_1, \dots, v_d of V , we may regard A_\bullet as a subsimplicial complex of \mathcal{BT}_\bullet using the map ι_{v_1, \dots, v_d} . This simplicial complex is denoted by $A_{v, \bullet} = A_{v_1, \dots, v_d, \bullet}$. Let $\overline{A_{v, \bullet}}$ denote the closure of $|A_{v, \bullet}|$ in $|\mathcal{BT}_\bullet|$ and set $\partial \overline{A_{v, \bullet}} = \overline{A_{v, \bullet}} \setminus |A_{v, \bullet}|$.

Let Δ'_{d-1} denote the interior of Δ_{d-1} . Let $\varphi : \Delta'_{d-1} \xrightarrow{\cong} \mathbb{R}^d / \mathbb{R}(1, \dots, 1)$ be the homeomorphism given by $(t_0, \dots, t_{d-1}) \mapsto (1/t_0, \dots, 1/t_{d-1})$. Let $n > 0$ be an integer. We set $\widetilde{K}_n = \prod_{i=0}^d [0, n] \subseteq \mathbb{R}^d$, and let K_n denote the image of \widetilde{K}_n in $\mathbb{R}^d / \mathbb{R}(1, \dots, 1)$. Recall (Section 3.2.1) that the simplices of A_\bullet are defined using the set of vertices $A_0 = \mathbb{Z}^{\oplus d} / \mathbb{Z}(1, \dots, 1)$. We regard $A_0 \subset \mathbb{R}^{\oplus d} / \mathbb{R}(1, \dots, 1)$ using the natural inclusion $\mathbb{Z} \subset \mathbb{R}$. The set of those simplices of A_\bullet whose support is contained in the complement of the interior of K_n naturally forms a subsimplicial complex of A_\bullet . We call this simplicial complex $K_{n, \bullet}^c$. It is easy to see that $\cap_n K_{n, \bullet}^c = \emptyset$.

Consider the following map

(5.6)

$$\begin{aligned} H_{d-1}(\overline{|A_{v, \bullet}|}, \partial \overline{|A_{v, \bullet}|}) &\rightarrow (H^{d-1}(\overline{|A_{v, \bullet}|}, \partial \overline{|A_{v, \bullet}|}))^* \rightarrow (\varinjlim_n H^{d-1}(\overline{|A_{v, \bullet}|}, \partial \overline{|A_{v, \bullet}|} \cup |K_{n, \bullet}^c|))^* \\ &\xleftarrow{\cong} (\varinjlim_n H^{d-1}(|A_{v, \bullet}|, |K_{n, \bullet}^c|))^* \xleftarrow{\cong} (\varinjlim_n H^{d-1}(A_{v, \bullet}, K_{n, \bullet}^c))^* \xleftarrow{\cong} H_c^{d-1}(A_{v, \bullet})^* \xleftarrow{\cong} H_{d-1}^{\text{BM}}(A_{v, \bullet}) \end{aligned}$$

where the limit is over the nonnegative integers in each case. The first map is the canonical map $A \rightarrow A^{**}$ for $A = H_{d-1}(\overline{|A_{v, \bullet}|}, \partial \overline{|A_{v, \bullet}|})$. The second map is the dual of the limit of the pullback map at each stage. The third map is the dual of the limit of the excision isomorphism of cohomology. The fourth map is the dual of the limit of the isomorphisms between cellular cohomology and singular cohomology (see Section 2.2.6). The fifth map is the map obtained from the definitions in Section 2.2.2. It is an isomorphism since $\cap_n K_{n, \bullet}^c = \emptyset$. The sixth map is the duality isomorphism in the universal coefficient theorem (see Section 2.2.3).

Note that the image of the continuous map $s(v_1, \dots, v_d)$ of Section 5.4 is contained in $\overline{|A_{v, \bullet}|}$ and defines a class $[s(v_1, \dots, v_d)]$ in $H_{d-1}(\overline{|A_{v, \bullet}|}, \partial \overline{|A_{v, \bullet}|})$.

Lemma 18. *The following two elements in $H_{d-1}^{\text{BM}}(A_\bullet)$ coincide:*

- (1) *The image of the class of $s(v_1, \dots, v_d)$ by*

$$H_{d-1}(\overline{|A_{v, \bullet}|}, \partial \overline{|A_{v, \bullet}|}) \rightarrow H_{d-1}^{\text{BM}}(A_{v, \bullet}) \xrightarrow{\iota_{v_1, \dots, v_d}^{-1}} H_{d-1}^{\text{BM}}(A_\bullet),$$

where the first map is the map in the diagram (5.6).

- (2) *The class of β of Section 3.2.3 in $H_{d-1}^{\text{BM}}(A_\bullet)$.*

Proof. Let us describe the image of the class $[s(v_1, \dots, v_d)]$ in $(\varinjlim_n H^{d-1}(A_\bullet, K_{n, \bullet}^c))^*$. Take an element $h \in \varinjlim_n H^{d-1}(A_\bullet, K_{n, \bullet}^c)$. We may suppose it is represented by an element h_m of $H^{d-1}(A_\bullet, K_{m, \bullet}^c)$ for some m .

Consider the map

$$H_{d-1}(\overline{|A_\bullet|}, \partial \overline{|A_\bullet|}) \rightarrow H_{d-1}(\overline{|A_\bullet|}, \partial \overline{|A_\bullet|} \cup |K_{m, \bullet}^c|) \xleftarrow{\cong} H_{d-1}(|A_\bullet|, |K_{m, \bullet}^c|) \xleftarrow{\cong} H_{d-1}(A_\bullet, K_{m, \bullet}^c).$$

Let t_m denote the image of $[s(v_1, \dots, v_d)]$ via this map. Then the pairing of $[s(v_1, \dots, v_d)]$ with the element h is the pairing of h_m and t_m under the canonical pairing

$$H_{d-1}(A_\bullet, K_{m, \bullet}^c) \times H^{d-1}(A_\bullet, K_{m, \bullet}^c) \rightarrow \mathbb{Q}$$

between homology and cohomology.

Let us compute t_m . Given $s = (s_0, \dots, s_{d-1}) \in K_m$, take a representative $\widetilde{s} = (\widetilde{s}_0, \dots, \widetilde{s}_{d-1}) \in \mathbb{R}^d$ such that $-m \leq s_i \leq 0$ ($0 \leq i \leq d-1$), $\min_i s_i = -m$. We define a map $g_m : K_m \rightarrow \Delta_{d-1}$ by $s \mapsto (\widetilde{s}_0 / (\widetilde{s}_0 +$

$\cdots + \widetilde{s_{d-1}}, \dots, \widetilde{s_{d-1}}/(\widetilde{s_0} + \cdots + \widetilde{s_{d-1}})$). It is well-defined and is a homeomorphism. Let $f_m : \Delta_{d-1} \rightarrow \Delta_{d-1}$ be the composite $\Delta_{d-1} \xrightarrow{g_n^{-1}} K_m \subset \mathbb{R}^d/\mathbb{R}(1, \dots, 1) \xleftarrow{\cong} \Delta'_{d-1} \subset \Delta_{d-1}$. From the following lemma, it follows that the class of $s(v_1, \dots, v_d)$ equals the class of $s(v_1, \dots, v_d) \circ f_m$ in $H_{d-1}(\overline{[A_\bullet]}, \partial \overline{[A_\bullet]} \cup |K_{m,\bullet}^c|)$. Note that $s(v_1, \dots, v_d) \circ f_m$ defines a class in $H_{d-1}(|A_\bullet|, |K_{m,\bullet}^c|)$, hence this is the image of $[s(v_1, \dots, s_d)]$.

It is easy to check that the class of $s(v_1, \dots, v_d) \circ f_m$ is then represented by the chain $(\gamma_\nu) \in \prod_{\nu \in A'_{d-1}} \mathbb{Z}$ where $\gamma_\nu = \beta_\nu$ for $\nu \in K'_m$ (β_ν was defined in Section 3.2.3), and $\gamma_\nu = 0$ for $\nu \notin K'_m$. \square

Lemma 19. *Let X be a topological space and Y a subspace. Let $n \geq 1$. Let $\alpha : \Delta_r \rightarrow X$ be a continuous map such that $\alpha((\Delta_r \setminus \text{Im} f_n)) \subset Y$. Then α and $\alpha \circ f_n : \Delta_r \rightarrow X$ both define the same element in $H_r(X, Y; \mathbb{Z})$.*

Proof. Omitted. \square

5.7. Proof of Theorem 10.

Proof. By Lemma 18, it suffices to show that

- (1) the image of the class of $[A_{v_1, \dots, v_d; g_0, \dots, g_d}]$ in $H_{d-1}^{\text{BM}}(\Gamma \backslash \mathcal{BT}_\bullet)$, and
- (2) the image of the class of $s(v_1, \dots, v_d)$ via the composite map

$$H_{d-1}(\overline{[A_{v,\bullet}]}, \partial \overline{[A_{v,\bullet}]}) \rightarrow H_{d-1}^{\text{BM}}(A_{v,\bullet}) \rightarrow H_{d-1}^{\text{BM}}(\Gamma \backslash \mathcal{BT}_\bullet),$$

where the first map is the map in the diagram (5.6),

coincide.

The argument is similar to that in the proof of Lemma 18. We compare the two classes in

$$\varinjlim_L H^{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet|, |L|)^*$$

which appeared in the definition of the map (3) in diagram (5.1). Let h be an element in $H^{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet|, |L|)$. We need to compute the images of the two classes in

$$H_{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet|, |L|)$$

Since the class (1) is represented by the chain $s(v_1, \dots, v_d) \times [g_0, \dots, g_{d-1}]$, using the argument as in the proof of Lemma 18, we see that it is represented by the chain $s(v_1, \dots, v_d) \circ f_n$ for sufficiently large n in $H_{d-1}(\Gamma \backslash |\mathcal{BT}_\bullet|, |L|)$. Again, as we have seen in the proof of Lemma 18, this is nothing but the class of (2). \square

6. THE HOMOLOGY OF AN ARITHMETIC QUOTIENT

In this section, we compute the homology groups and the Borel-Moore homology groups of some arithmetic quotients of the Bruhat-Tits building and relate them to the space of automorphic forms. The aim of this section is to prove Proposition 28 below.

6.1. Identification of homology groups and the space of automorphic forms.

For an open compact subgroup $\mathbb{K} \subset \text{GL}_d(\mathbb{A}^\infty)$, we let $\widetilde{X}_{\text{GL}_d, \mathbb{K}, \bullet}$ denote the disjoint union $\widetilde{X}_{\text{GL}_d, \mathbb{K}, \bullet} = (\text{GL}_d(\mathbb{A}^\infty)/\mathbb{K}) \times \mathcal{BT}_\bullet$ of copies of the Bruhat-Tits building \mathcal{BT}_\bullet indexed by $\text{GL}_d(\mathbb{A}^\infty)/\mathbb{K}$. We often omit the subscript GL_d on $\widetilde{X}_{\text{GL}_d, \mathbb{K}, \bullet}$ when there is no fear of confusion. The group $\text{GL}_d(\mathbb{A})$ acts on the simplicial complex $\widetilde{X}_{\mathbb{K}, \bullet}$ from the left. We study the quotient $\text{GL}_d(F) \backslash \widetilde{X}_{\mathbb{K}, \bullet}$ of $\widetilde{X}_{\mathbb{K}, \bullet}$ by the subgroup $\text{GL}_d(F) \subset \text{GL}_d(\mathbb{A})$.

For $0 \leq i \leq d-1$, we let $X_{\mathbb{K}, i} = X_{\text{GL}_d, \mathbb{K}, i}$ denote the quotient $X_{\mathbb{K}, i} = \text{GL}_d(F) \backslash \widetilde{X}_{\text{GL}_d, \mathbb{K}, i}$. We set $J_{\mathbb{K}} = \text{GL}_d(F) \backslash \text{GL}_d(\mathbb{A}^\infty)/\mathbb{K}$. For each $j \in J_{\mathbb{K}}$, we choose an element $g_j \in \text{GL}_d(\mathbb{A}^\infty)$ in the double coset j and set $\Gamma_j = \text{GL}_d(F) \cap g_j \mathbb{K} g_j^{-1}$. Then the set $X_{\mathbb{K}, i}$ is isomorphic to the disjoint union $\coprod_j \Gamma_j \backslash \mathcal{BT}_i$. For each j , the group $\Gamma_j \subset \text{GL}_d(F)$ is an arithmetic group as defined in Section 4.1. It follows that the tuple $X_{\mathbb{K}, \bullet} = (X_{\mathbb{K}, i})_{0 \leq i \leq d-1}$ forms a simplicial complex which is isomorphic to the disjoint union $\coprod_{j \in J_{\mathbb{K}}} \Gamma_j \backslash \mathcal{BT}_\bullet$.

Since the simplicial complex $\widetilde{X}_{\text{GL}_d, \mathbb{K}, \bullet}$ is locally finite, it follows that the simplicial complex $X_{\mathbb{K}, \bullet}$ is locally finite. Hence for an abelian group M , we may consider the cohomology groups with compact support $H_c^*(X_{\mathbb{K}, \bullet}, M)$ (resp. the Borel-Moore homology groups $H_*^{\text{BM}}(X_{\mathbb{K}, \bullet}, M)$) of the simplicial complex $X_{\mathbb{K}, \bullet}$.

Since the simplicial complex $X_{\mathbb{K}, \bullet}$ has no i -simplex for $i \geq d$ as was remarked in Section 3.1.4, it follows that the map

$$H_{d-1}(X_{\mathbb{K}, \bullet}, M) \rightarrow H_{d-1}^{\text{BM}}(X_{\mathbb{K}, \bullet}, M)$$

is injective for any abelian group M . We regard $H_{d-1}(X_{\mathbb{K}, \bullet}, M)$ as a subgroup of $H_{d-1}^{\text{BM}}(X_{\mathbb{K}, \bullet}, M)$.

6.1.1. Let St_d denote the Steinberg representation as defined, for example, in [La1, p.193]. It is defined with coefficients in \mathbb{C} , but it can also be defined with coefficients in \mathbb{Q} in a similar manner. We let St_d denote the corresponding representation.

Lemma 20. *For a \mathbb{Q} -vector space M , there is a canonical, $\text{GL}_d(F_\infty)$ -equivariant isomorphism between the module of M -valued harmonic $(d-1)$ -cochains and the module $\text{Hom}_{\mathbb{Q}}(\text{St}_d, M)$.*

Proof. By definition, the module of M -valued harmonic $(d-1)$ -cochains is identified with $\text{Hom}(H_c^{d-1}(\mathcal{BT}_\bullet, \mathbb{Q}), M)$. It is shown in [Bo, 6.2.6.4] that St_d (with \mathbb{C} -coefficient) is canonically isomorphic to $H_c^{d-1}(\mathcal{BT}_\bullet, \mathbb{C})$ as a representation of $\text{GL}_d(F_\infty)$. One can check that this map is defined over \mathbb{Q} . This proves the claim. \square

6.1.2. We let $\widetilde{\mathcal{BT}}_{j,*}$ denote the quotient $\widetilde{\mathcal{BT}}_j/F_\infty^\times$. This set is identified with the set of pairs (σ, v) with $\sigma \in \mathcal{BT}_j$ and $v \in \mathcal{BT}_0$ a vertex of σ , which we call a pointed j -simplex. Here the element $(L_i)_{i \in \mathbb{Z}} \bmod K^\times$ of $\widetilde{\mathcal{BT}}_j/K^\times$ corresponds to the pair $((L_i)_{i \in \mathbb{Z}}, L_0)$ via this identification.

We identify the set $\widetilde{\mathcal{BT}}_0$ with the coset $\text{GL}_d(K)/\text{GL}_d(\mathcal{O})$ by associating to an element $g \in \text{GL}_d(K)/\text{GL}_d(\mathcal{O})$ the lattice $\mathcal{O}_V g^{-1}$. Let $\mathcal{I} = \{(a_{ij}) \in \text{GL}_d(\mathcal{O}) \mid a_{ij} \bmod \varpi = 0 \text{ if } i > j\}$ be the Iwahori subgroup. Similarly, we identify the set $\widetilde{\mathcal{BT}}_{d-1}$ with the coset $\text{GL}_d(K)/\mathcal{I}$ by associating to an element $g \in \text{GL}_d(K)/\mathcal{I}$ the chain of lattices $(L_i)_{i \in \mathbb{Z}}$ characterized by $L_i = \mathcal{O}_V \Pi_i g^{-1}$ for $i = 0, \dots, d$. Here, for $i = 0, \dots, d$, we let Π_i denote the diagonal $d \times d$ matrix $\Pi_i = \text{diag}(\varpi, \dots, \varpi, 1, \dots, 1)$ with ϖ appearing i times and 1 appearing $d-i$ times.

Let M be a \mathbb{Q} -vector space. Let $\mathcal{C}^{\mathbb{K}}(M)$ denote the (\mathbb{Q} -vector) space of locally constant M -valued functions on $\text{GL}_d(F) \backslash \text{GL}_d(\mathbb{A}) / (\mathbb{K} \times F_\infty^\times)$. Let $\mathcal{C}_c^{\mathbb{K}}(M) \subset \mathcal{C}^{\mathbb{K}}(M)$ denote the subspace of compactly supported functions.

Lemma 21. (1) *There is a canonical isomorphism*

$$H_{d-1}^{\text{BM}}(X_{\mathbb{K}, \bullet}, M) \cong \text{Hom}_{\text{GL}_d(F_\infty)}(\text{St}_d, \mathcal{C}^{\mathbb{K}}(M)),$$

where $\mathcal{C}^{\mathbb{K}}(M)$ denotes the space of locally constant M -valued functions on $\text{GL}_d(F) \backslash \text{GL}_d(\mathbb{A}) / (\mathbb{K} \times F_\infty^\times)$.

(2) *Let $v \in \text{St}_d^{\mathcal{I}}$ be a non-zero Iwahori-spherical vector. Then the image of the evaluation map*

$$\begin{aligned} & \text{Hom}_{\text{GL}_d(F_\infty)}(\text{St}_d, \mathcal{C}^{\mathbb{K}}(M)) \\ & \rightarrow \text{Map}(\text{GL}_d(F) \backslash \text{GL}_d(\mathbb{A}) / (\mathbb{K} \times F_\infty^\times \mathcal{I}), M) \end{aligned}$$

at v is identified with the image of the map

$$\begin{aligned} H_{d-1}^{\text{BM}}(X_{\mathbb{K}, \bullet}, M) & \rightarrow \text{Map}(\text{GL}_d(F) \backslash (\text{GL}_d(\mathbb{A}^\infty) / \mathbb{K} \times \mathcal{BT}_{d-1,*}), M) \\ & \cong \text{Map}(\text{GL}_d(F) \backslash \text{GL}_d(\mathbb{A}) / (\mathbb{K} \times F_\infty^\times \mathcal{I}), M). \end{aligned}$$

Proof. For a \mathbb{C} -vector space M , (1) is proved in [Ko-Ya, Section 5.2.3], and (2) is [Ko-Ya, Corollary 5.7]. The proofs and the argument in loc. cit. work for a \mathbb{Q} -vector space M as well. \square

Corollary 22. *Under the isomorphism in (1), the subspace*

$$H_{d-1}(X_{\mathbb{K}, \bullet}, M) \subset H_{d-1}^{\text{BM}}(X_{\mathbb{K}, \bullet}, M)$$

corresponds to the subspace

$$\text{Hom}_{\text{GL}_d(F_\infty)}(\text{St}_d, \mathcal{C}_c^{\mathbb{K}}(M)) \subset \text{Hom}_{\text{GL}_d(F_\infty)}(\text{St}_d, \mathcal{C}^{\mathbb{K}}(M)).$$

Proof. This follows from Lemma 21 (2) and the definition of the homology group $H_{d-1}(X_{\mathbb{K}, \bullet}, M)$. \square

6.2. Pull-back maps for homology groups.

Let $\mathbb{K}, \mathbb{K}' \subset \text{GL}_d(\mathbb{A}^\infty)$ be open compact subgroups with $\mathbb{K}' \subset \mathbb{K}$. We denote by $f_{\mathbb{K}', \mathbb{K}}$ the natural projection map $X_{\mathbb{K}', i} \rightarrow X_{\mathbb{K}, i}$. Since \mathbb{K}' is a subgroup of \mathbb{K} of finite index, it follows that for any i with $0 \leq i \leq d-1$ and for any i -simplex $\sigma \in X_{\mathbb{K}, i}$, the inverse image of σ under the map $f_{\mathbb{K}', \mathbb{K}}$ is a finite set. Let i be an integer with $0 \leq i \leq d-1$ and let $\sigma' \in X_{\mathbb{K}', i}$. Let σ denote the image of σ' under the map $f_{\mathbb{K}', \mathbb{K}}$. Let us choose an i -simplex $\tilde{\sigma}'$ of $\tilde{X}_{\mathbb{K}', \bullet}$ which is sent to σ' under the projection map $\tilde{X}_{\mathbb{K}', \bullet} \rightarrow X_{\mathbb{K}', \bullet}$. Let $\tilde{\sigma}$ denote the image of $\tilde{\sigma}'$ under the map $\tilde{X}_{\mathbb{K}', i} \rightarrow \tilde{X}_{\mathbb{K}, i}$. We let

$$\Gamma_{\tilde{\sigma}'} = \{\gamma \in \text{GL}_d(F) \mid \gamma \tilde{\sigma}' = \tilde{\sigma}'\}$$

and

$$\Gamma_{\tilde{\sigma}} = \{\gamma \in \text{GL}_d(F) \mid \gamma \tilde{\sigma} = \tilde{\sigma}\}$$

denote the isotropy group of $\tilde{\sigma}'$ and $\tilde{\sigma}$, respectively.

The following lemma can be checked easily.

Lemma 23. *Let the notation be as above.*

- (1) *The group $\Gamma_{\tilde{\sigma}}$ is a finite group and the group $\Gamma_{\tilde{\sigma}'}$ is a subgroup of $\Gamma_{\tilde{\sigma}}$.*
- (2) *The isomorphism class of the group $\Gamma_{\tilde{\sigma}'}$ (resp. $\Gamma_{\tilde{\sigma}}$) depends only on σ' (resp. σ) and does not depend on the choice of $\tilde{\sigma}'$.*

The lemma above shows in particular that the index $[\Gamma_{\tilde{\sigma}} : \Gamma_{\tilde{\sigma}'}]$ is finite and depends only on σ' and $f_{\mathbb{K}', \mathbb{K}}$. We denote this index by $e_{\mathbb{K}', \mathbb{K}}(\sigma')$ and call it the ramification index of $f_{\mathbb{K}', \mathbb{K}}$ at σ' .

Let M be an abelian group. Let i be an integer with $0 \leq i \leq d$. We set $X'_{\mathbb{K}, i} = \coprod_{\sigma \in X_{\mathbb{K}, i}} O(\sigma)$. The map $f_{\mathbb{K}', \mathbb{K}} : X_{\mathbb{K}', \bullet} \rightarrow X_{\mathbb{K}, \bullet}$ induces a map $X'_{\mathbb{K}', i} \rightarrow X'_{\mathbb{K}, i}$ which we denote also by $f_{\mathbb{K}', \mathbb{K}}$. Let $m = (m_{\nu})_{\nu \in X'_{\mathbb{K}, i}}$ be an element of the $\{\pm 1\}$ -module $\prod_{\nu \in X'_{\mathbb{K}, i}} M$. We define the element $f_{\mathbb{K}', \mathbb{K}}^*(m)$ in $\prod_{\nu \in X'_{\mathbb{K}', i}} M$ to be

$$f_{\mathbb{K}', \mathbb{K}}^*(m) = (m'_{\nu'})_{\nu' \in X'_{\mathbb{K}', i}}$$

where for $\nu' \in O(\sigma') \subset X'_{\mathbb{K}', i}$, the element $m'_{\nu'} \in M$ is given by $m'_{\nu'} = e_{\mathbb{K}', \mathbb{K}}(\sigma') m_{f_{\mathbb{K}', \mathbb{K}}(\nu')}$. The following lemma can be checked easily.

Lemma 24. *Let the notation be as above.*

- (1) *The map $f_{\mathbb{K}', \mathbb{K}}^* : \prod_{\nu \in X'_{\mathbb{K}, i}} M \rightarrow \prod_{\nu' \in X'_{\mathbb{K}', i}} M$ is a homomorphism of $\{\pm 1\}$ -modules.*
- (2) *The map $f_{\mathbb{K}', \mathbb{K}}^* : \prod_{\nu \in X'_{\mathbb{K}, i}} M \rightarrow \prod_{\nu' \in X'_{\mathbb{K}', i}} M$ sends an element in the subgroup $\bigoplus_{\nu \in X'_{\mathbb{K}, i}} M \subset \prod_{\nu \in X'_{\mathbb{K}, i}} M$ to an element in $\bigoplus_{\nu' \in X'_{\mathbb{K}', i}} M$.*
- (3) *For $1 \leq i \leq d-1$, the diagrams*

$$\begin{array}{ccc} \prod_{\nu \in X'_{\mathbb{K}, i}} M & \xrightarrow{\tilde{\partial}_{i, \Pi}} & \prod_{\nu \in X'_{\mathbb{K}, i-1}} M \\ f_{\mathbb{K}', \mathbb{K}}^* \downarrow & & \downarrow f_{\mathbb{K}', \mathbb{K}}^* \\ \prod_{\nu' \in X'_{\mathbb{K}', i}} M & \xrightarrow{\tilde{\partial}_{i, \Pi}} & \prod_{\nu' \in X'_{\mathbb{K}', i-1}} M \end{array}$$

and

$$\begin{array}{ccc} \bigoplus_{\nu \in X'_{\mathbb{K}, i}} M & \xrightarrow{\tilde{\partial}_{i, \oplus}} & \bigoplus_{\nu \in X'_{\mathbb{K}, i-1}} M \\ f_{\mathbb{K}', \mathbb{K}}^* \downarrow & & \downarrow f_{\mathbb{K}', \mathbb{K}}^* \\ \bigoplus_{\nu' \in X'_{\mathbb{K}', i}} M & \xrightarrow{\tilde{\partial}_{i, \oplus}} & \bigoplus_{\nu' \in X'_{\mathbb{K}', i-1}} M \end{array}$$

are commutative.

The lemma above shows that the map $f_{\mathbb{K}', \mathbb{K}}^*$ induces homomorphisms $H_*(X_{\mathbb{K}, \bullet}, M) \rightarrow H_*(X_{\mathbb{K}', \bullet}, M)$ and $H_*^{\text{BM}}(X_{\mathbb{K}, \bullet}, M) \rightarrow H_*^{\text{BM}}(X_{\mathbb{K}', \bullet}, M)$ of abelian groups. We denote these homomorphisms also by $f_{\mathbb{K}', \mathbb{K}}^*$. We remark here that in [Ko-Ya, p.561, 5.3.3], we implicitly use these pullback maps for the Borel-Moore homology.

The proof of the following lemma is straightforward and is left to the reader.

Lemma 25. *Let the notation be as above.*

- (1) *Suppose that \mathbb{K}' is a normal subgroup of \mathbb{K} . Then the homomorphism $f_{\mathbb{K}', \mathbb{K}}^*$ induces an isomorphism $H_*^{\text{BM}}(X_{\mathbb{K}, \bullet}, M) \cong H_*^{\text{BM}}(X_{\mathbb{K}', \bullet}, M)^{\mathbb{K}/\mathbb{K}'}$ and a similar statement holds for H_* .*
- (2) *Let M be a \mathbb{Q} -vector space. Then the diagrams*

$$\begin{array}{ccc} H_{d-1}^{\text{BM}}(X_{\mathbb{K}, \bullet}, M) & \xrightarrow{\cong} & \text{Hom}_{\text{GL}_d(F_{\infty})}(\text{St}_d, \mathcal{C}^{\mathbb{K}}(M)) \\ f_{\mathbb{K}', \mathbb{K}}^* \downarrow & & \downarrow \\ H_{d-1}^{\text{BM}}(X_{\mathbb{K}', \bullet}, M) & \xrightarrow{\cong} & \text{Hom}_{\text{GL}_d(F_{\infty})}(\text{St}_d, \mathcal{C}^{\mathbb{K}'}(M)) \end{array}$$

and

$$\begin{array}{ccc} H_{d-1}(X_{\mathbb{K}, \bullet}, M) & \xrightarrow{\cong} & \text{Hom}_{\text{GL}_d(F_{\infty})}(\text{St}_d, \mathcal{C}_c^{\mathbb{K}}(M)) \\ f_{\mathbb{K}', \mathbb{K}}^* \downarrow & & \downarrow \\ H_{d-1}(X_{\mathbb{K}', \bullet}, M) & \xrightarrow{\cong} & \text{Hom}_{\text{GL}_d(F_{\infty})}(\text{St}_d, \mathcal{C}_c^{\mathbb{K}'}(M)) \end{array}$$

are commutative. Here the horizontal arrows are the isomorphisms given in Lemma 21 and Corollary 22, and the right vertical arrows are the map induced by the quotient map $\text{GL}_d(F) \backslash \text{GL}_d(\mathbb{A}) / (\mathbb{K}' \times F_{\infty}^{\times}) \rightarrow \text{GL}_d(F) \backslash \text{GL}_d(\mathbb{A}) / (\mathbb{K} \times F_{\infty}^{\times})$.

6.3. The action of $\mathrm{GL}_d(\mathbb{A}^\infty)$ and admissibility. For $g \in \mathrm{GL}_d(\mathbb{A}^\infty)$, we let $\tilde{\xi}_g : \tilde{X}_{\mathbb{K}, \bullet} \xrightarrow{\cong} \tilde{X}_{g^{-1}\mathbb{K}g, \bullet}$ denote the isomorphism of simplicial complexes induced by the isomorphism $\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K} \xrightarrow{\cong} \mathrm{GL}_d(\mathbb{A}^\infty)/g^{-1}\mathbb{K}g$ which sends a coset $h\mathbb{K}$ to the coset $hg \cdot g^{-1}\mathbb{K}g$ and by the identity on \mathcal{BT}_\bullet . The isomorphism $\tilde{\xi}_g$ induces an isomorphism $\xi_g : X_{\mathbb{K}, \bullet} \xrightarrow{\cong} X_{g^{-1}\mathbb{K}g, \bullet}$ of simplicial complexes. For two elements $g, g' \in \mathrm{GL}_d(\mathbb{A}^\infty)$, we have $\xi_{gg'} = \xi_{g'} \circ \xi_g$.

For an abelian group M , we let $H_*(X_{\mathrm{lim}, \bullet}, M) = H_*(X_{\mathrm{GL}_d, \mathrm{lim}, \bullet}, M)$ and $H_*^{\mathrm{BM}}(X_{\mathrm{lim}, \bullet}, M) = H_*^{\mathrm{BM}}(X_{\mathrm{GL}_d, \mathrm{lim}, \bullet}, M)$ denote the inductive limits $\varinjlim_{\mathbb{K}} H_*(X_{\mathbb{K}, \bullet}, M)$ and $\varinjlim_{\mathbb{K}} H_*^{\mathrm{BM}}(X_{\mathbb{K}, \bullet}, M)$, respectively. Here the transition maps in the inductive limits are given by $f_{\mathbb{K}', \mathbb{K}}^*$. The isomorphisms ξ_g for $g \in \mathrm{GL}_d(\mathbb{A}^\infty)$ gives rise to a smooth action of the group $\mathrm{GL}_d(\mathbb{A}^\infty)$ on these inductive limits. If M is a torsion free abelian group, then for each compact open subgroup $\mathbb{K} \subset \mathrm{GL}_d(\mathbb{A}^\infty)$, the homomorphism $H_*(X_{\mathbb{K}, \bullet}, M) \rightarrow H_*(X_{\mathrm{lim}, \bullet}, M)$ is injective and its image is equal to the \mathbb{K} -invariant part $H_*(X_{\mathrm{lim}, \bullet}, M)^{\mathbb{K}}$ of $H_*(X_{\mathrm{lim}, \bullet}, M)$. Similar statement holds for H_*^{BM} .

Lemma 26. (1) *For any open compact subgroup $\mathbb{K} \subset \mathrm{GL}_d(\mathbb{A}^\infty)$, both $H_{d-1}^{\mathrm{BM}}(X_{\mathbb{K}, \bullet}, \mathbb{Q})$ and $H_{d-1}(X_{\mathbb{K}, \bullet}, \mathbb{Q})$ are finite dimensional.*
 (2) *The inductive limits $H_{d-1}(X_{\mathrm{lim}, \bullet}, \mathbb{Q})$ and $H_{d-1}^{\mathrm{BM}}(X_{\mathrm{lim}, \bullet}, \mathbb{Q})$ are admissible $\mathrm{GL}_d(\mathbb{A}^\infty)$ -modules.*

Proof. It follows from Lemma 21 that the \mathbb{Q} -vector space $H_{d-1}^{\mathrm{BM}}(X_{\mathbb{K}, \bullet}, \mathbb{Q})$ is isomorphic to $\mathrm{Hom}_{\mathrm{GL}_d(F_\infty)}(\mathrm{St}_d, \mathcal{C}^{\mathbb{K}}(\mathbb{Q}))$. Let \mathcal{H}_∞ denote the convolution algebra of locally constant, compactly supported \mathbb{Q} -valued functions on $\mathrm{GL}_d(F_\infty)$, with respect to a Haar measure on $\mathrm{GL}_d(F_\infty)$ such that the volume of $\mathrm{GL}_d(\mathcal{O}_\infty)$ is a rational number. We regard St_d as a left \mathcal{H}_∞ -module. Let $K_\infty \subset \mathrm{GL}_d(F_\infty)$ be a compact open subgroup such that the K_∞ -invariant part $\mathrm{St}_d^{K_\infty}$ is non-zero. Let us fix a non-zero vector $v \in \mathrm{St}_d^{K_\infty}$ and let $J \subset \mathcal{H}_\infty$ denote the set of elements $f \in \mathcal{H}_\infty$ such that $fv = 0$. Then J is an admissible left ideal of \mathcal{H}_∞ in the sense of [Bo-Ja, 5.5, p.199]. The map $\mathrm{Hom}_{\mathrm{GL}_d(F_\infty)}(\mathrm{St}_d, \mathcal{C}^{\mathbb{K}}(\mathbb{Q})) \rightarrow \mathcal{C}^{\mathbb{K}}(\mathbb{Q})$ which sends $\varphi : \mathrm{St}_d \rightarrow \mathcal{C}^{\mathbb{K}}(\mathbb{Q})$ to $\varphi(v)$ gives an isomorphism from the space $\mathrm{Hom}_{\mathrm{GL}_d(F_\infty)}(\mathrm{St}_d, \mathcal{C}^{\mathbb{K}}(\mathbb{Q}))$ to the space of \mathbb{Q} -valued functions on $\mathrm{GL}_d(F) \backslash \mathrm{GL}_d(\mathbb{A})$ which is right invariant under $\mathbb{K} \times K_\infty$ and is annihilated by J . Hence it follows from [Bo-Ja, 5.6. THEOREM, p.199] that the space $\mathrm{Hom}_{\mathrm{GL}_d(F_\infty)}(\mathrm{St}_d, \mathcal{C}^{\mathbb{K}}(\mathbb{Q}))$ is finite dimensional. Therefore $H_{d-1}^{\mathrm{BM}}(X_{\mathbb{K}, \bullet}, \mathbb{Q})$ is finite dimensional. The space $H_{d-1}(X_{\mathbb{K}, \bullet}, \mathbb{Q})$ is finite dimensional since it is a subspace of $H_{d-1}^{\mathrm{BM}}(X_{\mathbb{K}, \bullet}, \mathbb{Q})$. This proves the claim (1).

For each compact open subgroup $\mathbb{K} \subset \mathrm{GL}_d(\mathbb{A}^\infty)$, the maps $H_{d-1}^{\mathrm{BM}}(X_{\mathbb{K}, \bullet}, \mathbb{Q}) \rightarrow H_{d-1}^{\mathrm{BM}}(X_{\mathrm{lim}, \bullet}, \mathbb{Q})^{\mathbb{K}}$ and $H_{d-1}(X_{\mathbb{K}, \bullet}, \mathbb{Q}) \rightarrow H_{d-1}(X_{\mathrm{lim}, \bullet}, \mathbb{Q})^{\mathbb{K}}$ are isomorphisms. Hence the claim (2) follows from the claim (1). \square

6.4. The homology.

Proposition 27. *As a representation of $\mathrm{GL}_d(\mathbb{A}^\infty)$, the inductive limit $H_{d-1}(X_{\mathrm{lim}, \bullet}, \mathbb{C})$ is isomorphic to the direct sum*

$$H_{d-1}(X_{\mathrm{lim}, \bullet}, \mathbb{C}) \cong \bigoplus_{\pi} \pi^\infty,$$

where $\pi = \pi^\infty \otimes \pi_\infty$ runs over (the isomorphism classes of) the irreducible cuspidal automorphic representations of $\mathrm{GL}_d(\mathbb{A})$ such that π_∞ is isomorphic to the Steinberg representation of $\mathrm{GL}_d(F_\infty)$.

Proof. Let $\mathcal{C}(\mathbb{C})$ be the space of locally constant, compactly supported \mathbb{C} -valued functions on $\mathrm{GL}_d(F) \backslash \mathrm{GL}_d(\mathbb{A}) / F_\infty^\times$. It follows from Corollary 22 that the isomorphism in Lemma 21 (1) induces a $\mathrm{GL}_d(\mathbb{A}) = \mathrm{GL}_d(\mathbb{A}^\infty) \times \mathrm{GL}_d(F_\infty)$ -equivariant homomorphism

$$\iota : H_{d-1}(X_{\mathrm{lim}, \bullet}, \mathbb{C}) \otimes_{\mathbb{Q}} \mathrm{St}_d \rightarrow \mathcal{C}(\mathbb{C}).$$

We denote by \mathcal{A} the image of the homomorphism ι . It follows from Corollary 22 that the map $H_{d-1}(X_{\mathrm{lim}, \bullet}, \mathbb{C}) \otimes_{\mathbb{Q}} \mathrm{St}_d^{\mathcal{I}} \rightarrow \mathcal{A}^{\mathcal{I}}$ is an isomorphism. We prove that $\mathcal{A}^{\mathcal{I}}$ is isomorphic to the right hand side of the desired isomorphism. Since $\mathcal{C}(\mathbb{C})$ consists of compactly supported functions, $\mathcal{C}(\mathbb{C})$ can be regarded as a subspace of $L^2(\mathrm{GL}_d(F) \backslash \mathrm{GL}_d(\mathbb{A}) / F_\infty^\times)$. It follows from Lemma 26 (2) that $H_{d-1}(X_{\mathrm{lim}, \bullet}, \mathbb{C}) \otimes_{\mathbb{Q}} \mathrm{St}_d$ is an admissible representation of $\mathrm{GL}_d(\mathbb{A})$. Hence \mathcal{A} is also an admissible representation of $\mathrm{GL}_d(\mathbb{A})$. Since \mathcal{A} is an admissible subrepresentation of $L^2(\mathrm{GL}_d(F) \backslash \mathrm{GL}_d(\mathbb{A}) / F_\infty^\times)$, it follows that \mathcal{A} is contained in a discrete spectrum of $L^2(\mathrm{GL}_d(F) \backslash \mathrm{GL}_d(\mathbb{A}) / F_\infty^\times)$ and is a direct sum of irreducible admissible representations. Let $\pi \subset \mathcal{A}$ be an irreducible subrepresentation. It follows from the construction of \mathcal{A} that the component π_∞ at ∞ of π is isomorphic to the Steinberg representation St_d . It follows from the classification ([Mo-Wa, p.606, Théorème]) of the discrete spectrum of $L^2(\mathrm{GL}_d(F) \backslash \mathrm{GL}_d(\mathbb{A}) / F_\infty^\times)$ that π does not belong to the residual spectrum. Hence π is an irreducible cuspidal automorphic representation. It follows from the multiplicity one theorem that \mathcal{A} is isomorphic to the direct sum of the irreducible subrepresentations of \mathcal{A} . Hence to prove the claim, it suffices to show that any cuspidal irreducible automorphic subrepresentation π of $L^2(\mathrm{GL}_d(F) \backslash \mathrm{GL}_d(\mathbb{A}) / F_\infty^\times)$ whose component π_∞ at ∞ is isomorphic to St_d is contained in \mathcal{A} . It is essentially proved in [Har, Theorem 1.2.1] (cf. [La2, p.16]) that the support of a cusp form on $\mathrm{GL}_d(\mathbb{A})$ is compact modulo center. It follows that

π is a subspace of $\mathcal{C}(\mathbb{C})$. Let us write $\pi = \pi^\infty \otimes \pi_\infty$. Since π_∞ is isomorphic to St_d , there exists, for any vector $v \in \pi^\infty$, a $\text{GL}_d(F_\infty)$ -equivariant homomorphism $\text{St}_d \rightarrow \mathcal{C}(\mathbb{C})$ whose image contains $\mathbb{Q}v \otimes \text{St}_d$. Hence it follows from Corollary 22 that $\mathbb{Q}v \otimes \text{St}_d$ is contained in \mathcal{A} . Therefore π is a subrepresentation of \mathcal{A} . This proves the claim. \square

Remark 6.1. We can prove Proposition 27 also by using the argument in [Har]: it follows that $H_{d-1}(X_{\text{lim}, \bullet}, \mathbb{C})$ is isomorphic to the subspace H of $L^2(\text{GL}_d(F)F_\infty^\times \backslash \text{GL}_d(\mathbb{A}))$ spanned by the subrepresentations whose component at ∞ is isomorphic to the Steinberg representation. Then the classification [Mo-Wa, p.606, Théorème] shows that any constituent of H is an irreducible cuspidal automorphic representation.

7. THE BOREL-MOORE HOMOLOGY OF AN ARITHMETIC QUOTIENT

The goal of this section is to prove the following theorem.

Theorem 28. *Let $\pi = \pi^\infty \otimes \pi_\infty$ be an irreducible smooth representation of $\text{GL}_d(\mathbb{A})$ such that π^∞ appears as a subquotient of $H_{d-1}^{\text{BM}}(X_{\text{lim}, \bullet}, \mathbb{C})$. Then there exist an integer $r \geq 1$, a partition $d = d_1 + \dots + d_r$ of d , and irreducible cuspidal automorphic representations π_i of $\text{GL}_{d_i}(\mathbb{A})$ for $i = 1, \dots, r$ which satisfy the following properties:*

- For each i with $0 \leq i \leq r$, the component $\pi_{i, \infty}$ at ∞ of π_i is isomorphic to the Steinberg representation of $\text{GL}_{d_i}(F_\infty)$.
- Let us write $\pi_i = \pi_i^\infty \otimes \pi_{i, \infty}$. Let $P \subset \text{GL}_d$ denote the standard parabolic subgroup corresponding to the partition $d = d_1 + \dots + d_r$. Then π^∞ is isomorphic to a subquotient of the unnormalized parabolic induction $\text{Ind}_{P(\mathbb{A}^\infty)}^{\text{GL}_d(\mathbb{A}^\infty)} \pi_1^\infty \otimes \dots \otimes \pi_r^\infty$.

Moreover for any subquotient H of $H_{d-1}^{\text{BM}}(X_{\text{lim}, \bullet}, \mathbb{C})$ which is of finite length as a representation of $\text{GL}_d(\mathbb{A}^\infty)$, the multiplicity of π in H is at most one.

Remark 7.1. Any open compact subgroup of $\text{GL}_d(\mathbb{A}^\infty)$ is conjugate to an open subgroup of $\text{GL}_d(\hat{A})$. The set of the open subgroups of $\text{GL}_d(\hat{A})$ is cofinal in the inductive system of all open compact subgroups of $\text{GL}_d(\mathbb{A}^\infty)$. Therefore, to prove Theorem 28, we may without loss of generality assume that the group \mathbb{K} is contained in $\text{GL}_d(\hat{A})$, and we may replace the inductive limit $\varinjlim_{\mathbb{K}}$ in the definition of $H_{d-1}^{\text{BM}}(X_{\text{lim}, \bullet}, M)$ and $H_{d-1}(X_{\text{lim}, \bullet}, M)$ with the inductive limit $\varinjlim_{\mathbb{K} \subset \text{GL}_d(\hat{A})}$.

From now on until the end of this section, we exclusively deal with the subgroups $\mathbb{K} \subset \text{GL}_d(\mathbb{A}^\infty)$ contained in $\text{GL}_d(\hat{A})$. The notation $\varinjlim_{\mathbb{K}}$ henceforth means the inductive limit $\varinjlim_{\mathbb{K} \subset \text{GL}_d(\hat{A})}$.

7.1. Chains of locally free \mathcal{O}_C -modules.

In this paragraph, we introduce some terminology for locally free \mathcal{O}_C -modules of rank d and then describe the sets of simplices of $X_{\mathbb{K}, \bullet}$ in terms of chains of locally free \mathcal{O}_C -modules of rank d . The terminology here will be used in our proof of Theorem 28.

Let $\eta : \text{Spec } F \rightarrow C$ denote the generic point of C . For each $g \in \text{GL}_d(\mathbb{A}^\infty)$ and an \mathcal{O}_∞ -lattice $L_\infty \subset \mathcal{O}_\infty^{\oplus d}$, we denote by $\mathcal{F}[g, L_\infty]$ the \mathcal{O}_C -submodule of $\eta_* F^{\oplus d}$ characterized by the following properties:

- $\mathcal{F}[g, L_\infty]$ is a locally free \mathcal{O}_C -module of rank d .
- $\Gamma(\text{Spec } A, \mathcal{F}[g, L_\infty])$ is equal to the A -submodule $\hat{A}^{\oplus d} g^{-1} \cap F^{\oplus d}$ of $F^{\oplus d} = \Gamma(\text{Spec } A, \eta_* F^{\oplus d})$.
- Let ι_∞ denote the morphism $\text{Spec } \mathcal{O}_\infty \rightarrow C$. Then $\Gamma(\text{Spec } \mathcal{O}_\infty, \iota_\infty^* \mathcal{F}[g, L_\infty])$ is equal to the \mathcal{O}_∞ -submodule L_∞ of $F_\infty^{\oplus d} = \Gamma(\text{Spec } \mathcal{O}_\infty, \iota_\infty^* \eta_* F^{\oplus d})$.

Let \mathcal{F} be a locally free \mathcal{O}_C -modules of rank d . Let $I \subset A$ be a non-zero ideal. We regard the A -module A/I as a coherent \mathcal{O}_C -module of finite length. A level I -structure on \mathcal{F} is a surjective homomorphism $\mathcal{F} \rightarrow (A/I)^{\oplus d}$ of \mathcal{O}_C -modules. Let $\mathbb{K}_I^\infty \subset \text{GL}_d(\hat{A})$ be the kernel of the homomorphism $\text{GL}_d(\hat{A}) \rightarrow \text{GL}_d(A/I)$. The group $\text{GL}_d(A/I) \cong \text{GL}_d(\hat{A})/\mathbb{K}_I^\infty$ acts from the left on the set of level I -structures on \mathcal{F} , via its left action on $(A/I)^{\oplus d}$. (We regard $(A/I)^{\oplus d}$ as an A -module of row vectors. The left action of $\text{GL}_d(A/I)$ on $(A/I)^{\oplus d}$ is described as $g \cdot b = bg^{-1}$ for $g \in \text{GL}_d(A/I)$, $b \in (A/I)^{\oplus d}$.) For a subgroup $\mathbb{K} \subset \text{GL}_d(\hat{A})$ containing \mathbb{K}_I^∞ , a level \mathbb{K} -structure on \mathcal{F} is a $\mathbb{K}/\mathbb{K}_I^\infty$ -orbit of level I -structures on \mathcal{F} . For an open subgroup $\mathbb{K} \subset \text{GL}_d(\hat{A})$, the set of level \mathbb{K} -structures on \mathcal{F} does not depend, up to canonical isomorphisms, on the choice of an ideal I with $\mathbb{K}_I^\infty \subset \mathbb{K}$.

Let $\mathbb{K} \subset \text{GL}_d(\hat{A})$ be an open subgroup. Let (g, σ) be an i -simplex of $\tilde{X}_{\mathbb{K}, \bullet}$. Take a chain $\dots \supsetneq L_{-1} \supsetneq L_0 \supsetneq L_1 \supsetneq \dots$ of \mathcal{O}_∞ -lattices of $F_\infty^{\oplus d}$ which represents σ . To (g, σ) we associate the chain $\dots \supsetneq \mathcal{F}[g, L_{-1}] \supsetneq \mathcal{F}[g, L_0] \supsetneq \mathcal{F}[g, L_1] \supsetneq \dots$ of \mathcal{O}_C -submodules of $\eta_* F^{\oplus d}$. Then the set of i -simplices in $\tilde{X}_{\mathbb{K}, \bullet}$ is identified with the set of the equivalence classes of chains $\dots \supsetneq \mathcal{F}_{-1} \supsetneq \mathcal{F}_0 \supsetneq \mathcal{F}_1 \supsetneq \dots$ of \mathcal{O}_∞ -lattices of locally free \mathcal{O}_C -submodules of rank d of $\eta_* \eta^* \mathcal{O}_C^{\oplus d}$ with a level \mathbb{K} -structure such that \mathcal{F}_{j-i-1} equals the twist $\mathcal{F}_j(\infty)$ as an \mathcal{O}_C -submodule of $\eta_* F^{\oplus d}$ with a level \mathbb{K} -structure for every $j \in \mathbb{Z}$. Two chains $\dots \supsetneq \mathcal{F}_{-1} \supsetneq \mathcal{F}_0 \supsetneq \mathcal{F}_1 \supsetneq \dots$

and $\cdots \supsetneq \mathcal{F}'_{-1} \supsetneq \mathcal{F}'_0 \supsetneq \mathcal{F}'_1 \supsetneq \cdots$ are equivalent if and only if there exists an integer l such that $\mathcal{F}_j = \mathcal{F}'_{j+l}$ as an \mathcal{O}_C -submodule of $\eta_* F^{\oplus d}$ with a level structure for every $j \in \mathbb{Z}$.

Let $g \in \mathrm{GL}_d(\mathbb{A}^\infty)$ and let L_∞ be an \mathcal{O}_∞ -lattice of $F_\infty^{\oplus d}$. For $\gamma \in \mathrm{GL}_d(F)$, the two \mathcal{O}_C -submodules $\mathcal{F}[g, L_\infty]$ and $\mathcal{F}[\gamma g, \gamma L_\infty]$ are isomorphic as \mathcal{O}_C -modules. The set of i -simplices in $X_{\mathbb{K}, \bullet}$ is identified with the set of the equivalence classes of chains $\cdots \hookrightarrow \mathcal{F}_1 \hookrightarrow \mathcal{F}_0 \hookrightarrow \mathcal{F}_{-1} \hookrightarrow \cdots$ of injective non-isomorphisms of locally free \mathcal{O}_C -modules of rank d with a level \mathbb{K} -structure such that the image of $\mathcal{F}_{j+i+1} \rightarrow \mathcal{F}_j$ equals the image of the canonical injection $\mathcal{F}_j(-\infty) \hookrightarrow \mathcal{F}_j$ for every $j \in \mathbb{Z}$. Two chains $\cdots \hookrightarrow \mathcal{F}_1 \hookrightarrow \mathcal{F}_0 \hookrightarrow \mathcal{F}_{-1} \hookrightarrow \cdots$ and $\cdots \hookrightarrow \mathcal{F}'_1 \hookrightarrow \mathcal{F}'_0 \hookrightarrow \mathcal{F}'_{-1} \hookrightarrow \cdots$ are equivalent if and only if there exists an integer l and an isomorphism $\mathcal{F}_j \cong \mathcal{F}'_{j+l}$ of \mathcal{O}_C -modules with level structures for every $j \in \mathbb{Z}$ such that the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_0 & \longrightarrow & \mathcal{F}_{-1} \longrightarrow \cdots \\ & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ \cdots & \longrightarrow & \mathcal{F}'_{l+1} & \longrightarrow & \mathcal{F}'_l & \longrightarrow & \mathcal{F}'_{l-1} \longrightarrow \cdots \end{array}$$

is commutative.

7.2. Harder-Narasimhan polygons. Let \mathcal{F} be a locally free \mathcal{O}_C -module of rank r . For an \mathcal{O}_C -submodule $\mathcal{F}' \subset \mathcal{F}$ (note that \mathcal{F}' is automatically locally free), we set $z_{\mathcal{F}}(\mathcal{F}') = (\mathrm{rank}(\mathcal{F}'), \deg(\mathcal{F}')) \in \mathbb{Q}^2$. It is known that there exists a unique convex, piecewise affine, affine on $[i-1, i]$ for $i = 1, \dots, r$, continuous function $p_{\mathcal{F}} : [0, r] \rightarrow \mathbb{R}$ on the interval $[0, r]$ such that the convex hull of the set $\{z_{\mathcal{F}}(\mathcal{F}') \mid \mathcal{F}' \subset \mathcal{F}\}$ in \mathbb{R}^2 equals $\{(x, y) \mid 0 \leq x \leq r, y \leq p_{\mathcal{F}}(x)\}$. We define the function $\Delta p_{\mathcal{F}} : \{1, \dots, d-1\} \rightarrow \mathbb{R}$ as $\Delta p_{\mathcal{F}}(i) = 2p_{\mathcal{F}}(i) - p_{\mathcal{F}}(i-1) - p_{\mathcal{F}}(i+1)$. Then $\Delta p_{\mathcal{F}}(i) \geq 0$ for all i . We note that for an invertible \mathcal{O}_C -module \mathcal{L} , $\Delta p_{\mathcal{F} \otimes \mathcal{L}}$ equals $\Delta p_{\mathcal{F}}$. The theory of Harder-Narasimhan filtration ([Har-Na]) implies that, if $i \in \mathrm{Supp}(\Delta p_{\mathcal{F}}) = \{i \mid \Delta p_{\mathcal{F}}(i) > 0\}$, then there exists a unique \mathcal{O}_C -submodule $\mathcal{F}' \subset \mathcal{F}$ satisfying $z_{\mathcal{F}}(\mathcal{F}') = (i, p_{\mathcal{F}}(i))$. We denote this \mathcal{O}_C -submodule \mathcal{F}' by $\mathcal{F}_{(i)}$. The submodule $\mathcal{F}_{(i)}$ has the following properties.

- If $i, j \in \mathrm{Supp}(\Delta p_{\mathcal{F}})$ with $i \leq j$, then $\mathcal{F}_{(i)} \subset \mathcal{F}_{(j)}$ and $\mathcal{F}_{(j)}/\mathcal{F}_{(i)}$ is locally free.
- If $i \in \mathrm{Supp}(\Delta p_{\mathcal{F}})$, then $p_{\mathcal{F}_{(i)}}(x) = p_{\mathcal{F}}(x)$ for $x \in [0, i]$ and $p_{\mathcal{F}/\mathcal{F}_{(i)}}(x-i) = p_{\mathcal{F}}(x) - \deg(\mathcal{F}_{(i)})$ for $x \in [i, r]$.

Lemma 29. *Let \mathcal{F} be a locally free \mathcal{O}_C -module of finite rank, and let $\mathcal{F}' \subset \mathcal{F}$ be a \mathcal{O}_C -submodule of the same rank. Then we have $0 \leq p_{\mathcal{F}}(i) - p_{\mathcal{F}'}(i) \leq \deg(\mathcal{F}) - \deg(\mathcal{F}')$ for $i = 1, \dots, \mathrm{rank}(\mathcal{F}) - 1$.*

Proof. Immediate from the definition of $p_{\mathcal{F}}$. □

7.3. In this paragraph, we state two propositions (Propositions 31 and 33). The proofs are given in Sections 28 and 31.

Given a subset $\mathcal{D} \subset \{1, \dots, d-1\}$ and a real number $\alpha > 0$, we define the simplicial subcomplex $X_{\mathbb{K}, \bullet}^{(\alpha), \mathcal{D}}$ of $X_{\mathbb{K}, \bullet}$ as follows: A simplex of $X_{\mathbb{K}, \bullet}$ belongs to $X_{\mathbb{K}, \bullet}^{(\alpha), \mathcal{D}}$ if and only if each of its vertices is represented by a locally free \mathcal{O}_C -module \mathcal{F} of rank d with a level \mathbb{K} -structure such that $\Delta p_{\mathcal{F}}(i) \geq \alpha$ holds for every $i \in \mathcal{D}$.

Let $X_{\mathbb{K}, \bullet}^{(\alpha)}$ denote the union $X_{\mathbb{K}, \bullet}^{(\alpha)} = \bigcup_{\mathcal{D} \neq \emptyset} X_{\mathbb{K}, \bullet}^{(\alpha), \mathcal{D}}$.

Lemma 30. *For any $\alpha > 0$, the set of the simplices in $X_{\mathbb{K}, \bullet}$ not belonging to $X_{\mathbb{K}, \bullet}^{(\alpha)}$ is finite.*

Proof. Let \mathcal{P} denote the set of continuous, convex functions $p' : [0, d] \rightarrow \mathbb{R}$ with $p'(0) = 0$ such that $p'(i) \in \mathbb{Z}$ and p' is affine on $[i-1, i]$ for $i = 1, \dots, d$. It is known that for any $r \geq 1$ and $f \in \mathbb{Z}$, there are only a finite number of isomorphism classes of semi-stable locally free \mathcal{O}_C -modules of rank r with degree f . Hence by the theory of Harder-Narasimhan filtration, for any $p' \in \mathcal{P}$, the set of the isomorphism classes of locally free \mathcal{O}_C -modules \mathcal{F} with $p_{\mathcal{F}} = p'$ is finite. Let us give an action of the group \mathbb{Z} on the set \mathcal{P} , by setting $(a \cdot p')(x) = p'(x) + a \deg(\infty)x$ for $a \in \mathbb{Z}$ and for $p' \in \mathcal{P}$. Then $p_{\mathcal{F}(a\infty)} = a \cdot p_{\mathcal{F}}$ for any $a \in \mathbb{Z}$ and for any locally free \mathcal{O}_C -module \mathcal{F} of rank d . For $\alpha > 0$ let $\mathcal{P}^{(\alpha)} \subset \mathcal{P}$ denote the set of functions $p' \in \mathcal{P}$ with $2p'(i) - p'(i-1) - p'(i+1) \leq \alpha$ for each $i \in \{1, \dots, d-1\}$. An elementary argument shows that the quotient $\mathcal{P}^{(\alpha)}/\mathbb{Z}$ is a finite set, whence the claim follows. □

Lemma 30 implies that $H_{d-1}^{\mathrm{BM}}(X_{\mathbb{K}, \bullet}, \mathbb{Q})$ is canonically isomorphic to the projective limit $\varprojlim_{\alpha > 0} H_{d-1}(X_{\mathbb{K}, \bullet}, X_{\mathbb{K}, \bullet}^{(\alpha)}; \mathbb{Q})$ and $H_c^{d-1}(X_{\mathbb{K}, \bullet}, \mathbb{Q})$ is canonically isomorphic to the inductive limit $\varinjlim_{\alpha > 0} H_c^{d-1}(X_{\mathbb{K}, \bullet}, X_{\mathbb{K}, \bullet}^{(\alpha)}; \mathbb{Q})$. Thus we have an exact sequence

$$(7.1) \quad \varinjlim_{\alpha > 0} H^{d-2}(X_{\mathbb{K}, \bullet}^{(\alpha)}, \mathbb{Q}) \rightarrow H_c^{d-1}(X_{\mathbb{K}, \bullet}, \mathbb{Q}) \rightarrow H^{d-1}(X_{\mathbb{K}, \bullet}, \mathbb{Q}) \rightarrow \varinjlim_{\alpha > 0} H^{d-1}(X_{\mathbb{K}, \bullet}^{(\alpha)}, \mathbb{Q}).$$

Proposition 31. *For $\alpha' \geq \alpha > (d-1)\deg(\infty)$, the homomorphisms $H^*(X_{\mathbb{K},\bullet}^{(\alpha)}, \mathbb{Q}) \rightarrow H^*(X_{\mathbb{K},\bullet}^{(\alpha')}, \mathbb{Q})$ are isomorphisms.*

We give a proof of Proposition 31 in Section 7.5.

Lemma 32. *For every $g \in \mathrm{GL}_d(\mathbb{A}^\infty)$ satisfying $g^{-1}\mathbb{K}g \subset \mathrm{GL}_d(\hat{A})$, there exists a real number $\beta_g \geq 0$ such that the isomorphism $\xi_g : X_{\mathbb{K},\bullet} \xrightarrow{\cong} X_{g^{-1}\mathbb{K}g,\bullet}$ sends $X_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}$ to $X_{g^{-1}\mathbb{K}g,\bullet}^{(\alpha-\beta_g),\mathcal{D}} \subset X_{g^{-1}\mathbb{K}g,\bullet}$ for all $\alpha > \beta_g$, for all open compact $\mathbb{K} \subset \mathrm{GL}_d(\mathbb{A}^\infty)$, and for all nonempty subset $\mathcal{D} \subset \{1, \dots, d-1\}$.*

Proof. Take two elements $a, b \in \mathbb{A}^{\infty \times} \cap \hat{A}$ such that both ag and bg^{-1} lie in $\mathrm{GL}_d(\mathbb{A}^\infty) \cap \mathrm{Mat}_d(\hat{A})$. Then for any $h \in \mathrm{GL}_d(\mathbb{A}^\infty)$ we have $a\hat{A}^{\oplus d}h^{-1} \subset \hat{A}^{\oplus d}g^{-1}h^{-1} \subset b^{-1}\hat{A}^{\oplus d}h^{-1}$. This implies that, for any vertex $x \in X_{\mathbb{K},0}$, if we take suitable representatives $\mathcal{F}_x, \mathcal{F}_{\xi_g(x)}$ of the equivalence classes of locally free \mathcal{O}_C -modules corresponding to $x, \xi_g(x)$, then there exists a sequence of injections $\mathcal{F}_x(-\mathrm{div}(a)) \hookrightarrow \mathcal{F}_{\xi_g(x)} \hookrightarrow \mathcal{F}_x(\mathrm{div}(b))$. Applying Lemma 29, we see that there exists a positive real number $m_g > 0$ not depending on x such that $|p_{\mathcal{F}_x}(i) - p_{\mathcal{F}_{\xi_g(x)}}(i)| < m_g$ for all i . Hence the claim follows. \square

Thus the group $\mathrm{GL}_d(\mathbb{A}^\infty)$ acts on $\varinjlim_{\mathbb{K}} \varinjlim_{\alpha > 0} H^*(X_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}, \mathbb{Q})$ in such a way that the exact sequence (7.1) is $\mathrm{GL}_d(\mathbb{A}^\infty)$ -equivariant.

We use a covering spectral sequence

$$(7.2) \quad E_1^{p,q} = \bigoplus_{\#\mathcal{D}=p+1} H^q(X_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}, \mathbb{Q}) \Rightarrow H^{p+q}(X_{\mathbb{K},\bullet}^{(\alpha)}, \mathbb{Q})$$

with respect to the covering $X_{\mathbb{K},\bullet}^{(\alpha)} = \bigcup_{1 \leq i \leq d-1} X_{\mathbb{K},\bullet}^{(\alpha),\{i\}}$ of $X_{\mathbb{K},\bullet}^{(\alpha)}$. For $\alpha' \geq \alpha > 0$, the inclusion $X_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}} \rightarrow X_{\mathbb{K},\bullet}^{(\alpha'),\mathcal{D}}$ induces a morphism of spectral sequences. Taking the inductive limit, we obtain the spectral sequence

$$E_1^{p,q} = \bigoplus_{\#\mathcal{D}=p+1} \varinjlim_{\alpha} H^q(X_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}, \mathbb{Q}) \Rightarrow \varinjlim_{\alpha} H^{p+q}(X_{\mathbb{K},\bullet}^{(\alpha)}, \mathbb{Q}).$$

For $g \in \mathrm{GL}_d(\mathbb{A}^\infty)$ satisfying $g^{-1}\mathbb{K}g \subset \mathrm{GL}_d(\hat{A})$, let β_g be as in Lemma 32. Then for $\alpha > \beta_g$ the isomorphism $\xi_g : X_{\mathbb{K},\bullet} \xrightarrow{\cong} X_{g^{-1}\mathbb{K}g,\bullet}$ induces a homomorphism from the spectral sequence (7.2) for $X_{\mathbb{K},\bullet}^{(\alpha)}$ to that for $X_{\mathbb{K},\bullet}^{(\alpha-\beta_g)}$. Passing to the inductive limit with respect to α and then passing to the inductive limit with respect to \mathbb{K} , we obtain the left action of the group $\mathrm{GL}_d(\mathbb{A}^\infty)$ on the spectral sequence

$$(7.3) \quad E_1^{p,q} = \bigoplus_{\#\mathcal{D}=p+1} \varinjlim_{\mathbb{K}} \varinjlim_{\alpha} H^q(X_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}, \mathbb{Q}) \Rightarrow \varinjlim_{\mathbb{K}} \varinjlim_{\alpha} H^{p+q}(X_{\mathbb{K},\bullet}^{(\alpha)}, \mathbb{Q}).$$

For a subset \mathcal{D} of $\{1, \dots, d-1\}$, we define the algebraic groups $P_{\mathcal{D}}, N_{\mathcal{D}}$ and $M_{\mathcal{D}}$ as follows. We write $\mathcal{D} = \{i_1, \dots, i_{r-1}\}$, with $i_0 = 0 < i_1 < \dots < i_{r-1} < i_r = d$ and set $d_j = i_j - i_{j-1}$ for $j = 1, \dots, r$. We define $P_{\mathcal{D}}, N_{\mathcal{D}}$ and $M_{\mathcal{D}}$ as the standard parabolic subgroup of GL_d of type (d_1, \dots, d_r) , the unipotent radical of $P_{\mathcal{D}}$, and the quotient group $P_{\mathcal{D}}/N_{\mathcal{D}}$ respectively. We identify the group $M_{\mathcal{D}}$ with $\mathrm{GL}_{d_1} \times \dots \times \mathrm{GL}_{d_r}$.

Proposition 33. *Let the notations be above. Then as a smooth $\mathrm{GL}_d(\mathbb{A}^\infty)$ -module, $\varinjlim_{\mathbb{K}} \varinjlim_{\alpha > 0} H^*(X_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}, \mathbb{Q})$ is isomorphic to*

$$\mathrm{Ind}_{P_{\mathcal{D}}(\mathbb{A}^\infty)}^{\mathrm{GL}_d(\mathbb{A}^\infty)} \bigotimes_{j=1}^r \varinjlim_{\mathbb{K}_j \subset \mathrm{GL}_{d_j}(\hat{A})} H^*(X_{\mathrm{GL}_{d_j}, \mathbb{K}_j, \bullet}, \mathbb{Q}),$$

where the group $P_{\mathcal{D}}(\mathbb{A}^\infty)$ acts on $\bigotimes_{j=1}^r \varinjlim_{\mathbb{K}_j \subset \mathrm{GL}_{d_j}(\hat{A})} H^*(X_{\mathrm{GL}_{d_j}, \mathbb{K}_j, \bullet}, \mathbb{Q})$ via the quotient $P_{\mathcal{D}}(\mathbb{A}^\infty) \rightarrow M_{\mathcal{D}}(\mathbb{A}^\infty) = \prod_j \mathrm{GL}_{d_j}(\mathbb{A}^\infty)$, and $\mathrm{Ind}_{P_{\mathcal{D}}(\mathbb{A}^\infty)}^{\mathrm{GL}_d(\mathbb{A}^\infty)}$ denotes the parabolic induction unnormalized by the modulus function.

The proof will be given in Section 7.6.

7.4. Proof of Theorem 28. Here we give a proof of Theorem 28 assuming Propositions 31 and 33.

Proof of Theorem 28. Let $\mathbb{K}, \mathbb{K}' \subset \mathrm{GL}_d(\mathbb{A}^\infty)$ be two compact open subgroups with $\mathbb{K}' \subset \mathbb{K}$. The pull-back morphism from the cochain complex of $X_{\mathbb{K},\bullet}$ to that of $X_{\mathbb{K}',\bullet}$ preserves the cochains with finite supports. Thus we have pull-back homomorphisms $H_c^*(X_{\mathbb{K},\bullet}, \mathbb{Q}) \rightarrow H_c^*(X_{\mathbb{K}',\bullet}, \mathbb{Q})$ which is compatible with the usual pull-back homomorphism $H^*(X_{\mathbb{K},\bullet}, \mathbb{Q}) \rightarrow H^*(X_{\mathbb{K}',\bullet}, \mathbb{Q})$. For an abelian group M , we let $H^*(X_{\mathrm{lim},\bullet}, M) = H^*(X_{\mathrm{GL}_d, \mathrm{lim}, \bullet}, M)$ and $H_c^*(X_{\mathrm{lim},\bullet}, M) = H_c^*(X_{\mathrm{GL}_d, \mathrm{lim}, \bullet}, M)$ denote the inductive limits $\varinjlim_{\mathbb{K}} H^*(X_{\mathbb{K},\bullet}, M)$ and $\varinjlim_{\mathbb{K}} H_c^*(X_{\mathbb{K},\bullet}, M)$, respectively. If M is a \mathbb{Q} -vector space, then for each compact open subgroup $\mathbb{K} \subset \mathrm{GL}_d(\mathbb{A}^\infty)$, the homomorphism $H^*(X_{\mathbb{K},\bullet}, M) \rightarrow H^*(X_{\mathrm{lim},\bullet}, M)$ is injective and its image is equal to the \mathbb{K} -invariant part $H^*(X_{\mathrm{lim},\bullet}, M)^{\mathbb{K}}$ of $H^*(X_{\mathrm{lim},\bullet}, M)$. Similar statement holds for H_c^* . It follows from

Lemma 26 that the inductive limits $H^{d-1}(X_{\lim, \bullet}, \mathbb{Q})$ and $H_c^{d-1}(X_{\lim, \bullet}, \mathbb{Q})$ are admissible $\mathrm{GL}_d(\mathbb{A}^\infty)$ -modules, and are isomorphic to the contragradient of $H_{d-1}(X_{\lim, \bullet}, \mathbb{Q})$ and $H_{d-1}^{\mathrm{BM}}(X_{\lim, \bullet}, \mathbb{Q})$, respectively. Since St_d is self-contragradient, it follows from the compatibility of the normalized parabolic inductions with taking contragradient that it suffice to prove that any irreducible subquotient of $H_c^{d-1}(X_{\lim, \bullet}, \mathbb{C})$ satisfies the properties in the statement of Theorem 28. Let π be an irreducible subquotient of $H_c^{d-1}(X_{\lim, \bullet}, \mathbb{C})$. Then Proposition 33 combined with the spectral sequence (7.3) shows that there exists a subset $\mathcal{D} \subset \{1, \dots, d-1\}$ such that π^∞ is isomorphic to a subquotient of $\mathrm{Ind}_{P_{\mathcal{D}}(\mathbb{A}^\infty)}^{\mathrm{GL}_d(\mathbb{A}^\infty)} \bigotimes_{j=1}^r \varinjlim_{\mathbb{K}_j} H^{d_j-1}(X_{\mathrm{GL}_{d_j}, \mathbb{K}_j, \bullet}, \mathbb{C})$. Here $r = \#\mathcal{D} + 1$, and $d_1, \dots, d_r \geq 1$ are the integers satisfying $\mathcal{D} = \{d_1, d_1 + d_2, \dots, d_1 + \dots + d_{r-1}\}$ and $d_1 + \dots + d_r = d$. By Proposition 27, π^∞ is isomorphic to a subquotient of the non- ∞ -component of the induced representation from $P_{\mathcal{D}}(\mathbb{A})$ to $\mathrm{GL}_d(\mathbb{A})$ of an irreducible cuspidal automorphic representation $\pi_1 \otimes \dots \otimes \pi_r$ of $M_{\mathcal{D}}(\mathbb{A})$ whose component at ∞ is isomorphic to the tensor product the Steinberg representations.

It remains to prove the claim of the multiplicity. The Ramanujan-Petersson conjecture proved by Lafforgue shows that each place v of F , the representation $\pi_{i,v}$ is tempered. Hence for almost all places v of F , the representation π_v of $\mathrm{GL}_d(F_v)$ is unramified and its associated Satake parameters $\alpha_{v,1}, \dots, \alpha_{v,d}$ have the following property: for each i with $1 \leq i \leq r$, exactly d_i parameters of $\alpha_{v,1}, \dots, \alpha_{v,d}$ have the complex absolute value $q_v^{a_i/2}$ where q_v denotes the cardinality of the residue field at v and $a_i = \sum_{i < j \leq r} d_j - \sum_{1 \leq j < i} d_j$. This shows that the subset \mathcal{D} is uniquely determined by π . It follows from the multiplicity one theorem and the strong multiplicity one theorem that the cuspidal automorphic representation $\pi_1 \otimes \dots \otimes \pi_r$ of $M_{\mathcal{D}}(\mathbb{A})$ is also uniquely determined by π . Hence it suffices to show that the representation $\mathrm{Ind}_{P_{\mathcal{D}}(F_v)}^{\mathrm{GL}_d(F_v)} \pi_{1,v} \otimes \dots \otimes \pi_{r,v}$ of $\mathrm{GL}_d(F)$ is of multiplicity free for every place v of F . For $1 \leq i \leq r$, let Δ_i denote the multiset of segments corresponding to the representation $\pi_{i,v} \otimes |\det(\cdot)|_v^{a_i/2}$ in the sense of [Ze]. We denote by Δ_i^t the Zelevinski dual of Δ_i . Let i_1, i_2 be integers with $1 \leq i_1 < i_2 \leq r$ and suppose that there exist a segment in $\Delta_{i_1}^t$ and a segment in $\Delta_{i_2}^t$ which are linked. Since $\pi_{i_1,v}$ and $\pi_{i_2,v}$ are tempered, it follows that $i_2 = i_1 + 1$ and that there exists a character χ of F_v^\times such that both $\pi_{i_1,v} \otimes \chi$ and $\pi_{i_2,v} \otimes \chi$ are the Steinberg representations. In this case the multiset $\Delta_{i_j}^t$ consists of a single segment for $j = 1, 2$ and the the unique segment in $\Delta_{i_1}^t$ and the unique segment in $\Delta_{i_2}^t$ are juxtaposed. Thus the claim is obtained by applying the formula in [Ze, 9.13, Proposition, p.201]. \square

7.5. Proof of Proposition 31. We need some preparation.

Lemma 34. *Let \mathcal{F} be a locally free \mathcal{O}_C -module of rank d . Let $\mathcal{F}' \subset \mathcal{F}$ be a n \mathcal{O}_C -submodule of the same rank. Suppose that $\Delta_{\mathcal{F}}(i) > \deg(\mathcal{F}) - \deg(\mathcal{F}')$. Then we have $\mathcal{F}'_{(i)} = \mathcal{F}_{(i)} \cap \mathcal{F}'$.*

Proof. It suffices to prove that $\mathcal{F}'_{(i)} \subset \mathcal{F}_{(i)}$. Assume otherwise. Let us consider the short exact sequence

$$0 \rightarrow \mathcal{F}'_{(i)} \cap \mathcal{F}_{(i)} \rightarrow \mathcal{F}'_{(i)} \rightarrow \mathcal{F}'_{(i)} / (\mathcal{F}'_{(i)} \cap \mathcal{F}_{(i)}) \rightarrow 0$$

Let r denote the rank of $\mathcal{F}'_{(i)} \cap \mathcal{F}_{(i)}$. By assumption, r is strictly smaller than i . Hence

$$\begin{aligned} \deg(\mathcal{F}'_{(i)}) &= \deg(\mathcal{F}'_{(i)} \cap \mathcal{F}_{(i)}) + \deg(\mathcal{F}'_{(i)} / (\mathcal{F}'_{(i)} \cap \mathcal{F}_{(i)})) \\ &\leq p_{\mathcal{F}}(r) + p_{\mathcal{F}/\mathcal{F}_{(i)}}(i-r) \\ &\leq p_{\mathcal{F}}(i) - (i-r)(p_{\mathcal{F}}(i) - p_{\mathcal{F}}(i-1)) + (i-r)(p_{\mathcal{F}}(i+1) - p_{\mathcal{F}}(i)) \\ &= \deg(\mathcal{F}_{(i)}) - (i-r)\Delta_{\mathcal{F}}(i) \\ &< \deg(\mathcal{F}_{(i)}) - (\deg(\mathcal{F}) - \deg(\mathcal{F}')). \end{aligned}$$

On the other hand, Lemma 29 shows that $\deg(\mathcal{F}_{(i)} \cap \mathcal{F}') \geq \deg(\mathcal{F}_{(i)}) - (\deg(\mathcal{F}) - \deg(\mathcal{F}'))$. This is a contradiction. \square

Let $\mathrm{Flag}_{\mathcal{D}}$ denote the set

$$\mathrm{Flag}_{\mathcal{D}} = \{f = [0 \subset V_1 \subset \dots \subset V_{r-1} \subset F^{\oplus d}] \mid \dim(V_j) = i_j\}$$

of flags in $F^{\oplus d}$.

Let $\tilde{X}_{\mathbb{K}, \bullet}^{(\alpha), \mathcal{D}}$ denote the inverse image of $X_{\mathbb{K}, \bullet}^{(\alpha), \mathcal{D}}$ by the morphism $\tilde{X}_{\mathbb{K}, \bullet} \rightarrow X_{\mathbb{K}, \bullet}$. For $f = [0 \subset V_1 \subset \dots \subset V_{r-1} \subset F^{\oplus d}] \in \mathrm{Flag}_{\mathcal{D}}$, let $\tilde{X}_{\mathbb{K}, \bullet}^{(\alpha), \mathcal{D}, f}$ denote the simplicial subcomplex of $\tilde{X}_{\mathbb{K}, \bullet}^{(\alpha), \mathcal{D}}$ consisting of the simplices in $\tilde{X}_{\mathbb{K}, \bullet}$ whose representative $\dots \supseteq \mathcal{F}_{-1} \supseteq \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \dots$ satisfies $\mathcal{F}_{l, (i_j)} = \mathcal{F}_l \cap \eta_* V_{i_j}$ for every $l \in \mathbb{Z}$, $j = 1, \dots, r-1$. Lemma 34 implies that, for $\alpha > (d-1)\deg(\infty)$, $\tilde{X}_{\mathbb{K}, \bullet}^{(\alpha), \mathcal{D}}$ is decomposed into a disjoint union $\tilde{X}_{\mathbb{K}, \bullet}^{(\alpha), \mathcal{D}} = \coprod_{f \in \mathrm{Flag}_{\mathcal{D}}} \tilde{X}_{\mathbb{K}, \bullet}^{(\alpha), \mathcal{D}, f}$. An argument similar to that in the proof of Lemma 32 shows that, for each $g \in \mathrm{GL}_d(\mathbb{A}^\infty)$ satisfying $g^{-1}\mathbb{K}g \subset \mathrm{GL}_d(\hat{A})$, there exists a real number $\beta'_g > \beta_g$ such that the isomorphism $\tilde{\xi}_g$ sends $\tilde{X}_{\mathbb{K}, \bullet}^{(\alpha), \mathcal{D}, f}$ to $\tilde{X}_{g\mathbb{K}g^{-1}, \bullet}^{(\alpha - \beta_g), \mathcal{D}, f} \subset \tilde{X}_{g\mathbb{K}g^{-1}, \bullet}^{(\alpha), \mathcal{D}}$ for $\alpha > \beta'_g$ and for any $f \in \mathrm{Flag}_{\mathcal{D}}$.

For $\gamma \in \mathrm{GL}_d(F)$, the action of γ on $\tilde{X}_{\mathbb{K},\bullet}$ sends $\tilde{X}_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D},f}$ bijectively to $\tilde{X}_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D},\gamma f}$. Let $f_0 = [0 \subset F^{\oplus i_1} \oplus \{0\}^{\oplus d-i_1} \subset \dots \subset F^{\oplus i_{r-1}} \oplus \{0\}^{\oplus d-i_{r-1}} \subset F^{\oplus d}] \in \mathrm{Flag}_{\mathcal{D}}$ be the standard flag. The group $\mathrm{GL}_d(F)$ acts transitively on $\mathrm{Flag}_{\mathcal{D}}$ and its stabilizer at f_0 equals $P_{\mathcal{D}}(F)$. Hence for $\alpha > (d-1)\deg(\infty)$,

$X_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}$ is isomorphic to the quotient $P_{\mathcal{D}}(F) \backslash \tilde{X}_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D},f_0}$.

For $g \in \mathrm{GL}_d(\mathbb{A}^\infty)$, we set

$$\tilde{Y}_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D},g} = \tilde{X}_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D},f_0} \cap (P_{\mathcal{D}}(\mathbb{A}^\infty)g / (g^{-1}P_{\mathcal{D}}(\mathbb{A}^\infty)g \cap \mathbb{K}) \times \mathcal{BT}_\bullet)$$

and $Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D},g} = P_{\mathcal{D}}(F) \backslash \tilde{Y}_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D},g}$. We omit the superscript g on $\tilde{Y}_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D},g}$ and $Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D},g}$ if $g = 1$. If we take a complete set $T \subset \mathrm{GL}_d(\mathbb{A}^\infty)$ of representatives of $P_{\mathcal{D}}(\mathbb{A}^\infty) \backslash \mathrm{GL}_d(\mathbb{A}^\infty)$, then we have $\tilde{X}_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D},f_0} = \coprod_{g \in T} \tilde{Y}_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D},g}$. For each $g \in \mathrm{GL}_d(\mathbb{A}^\infty)$ satisfying $g^{-1}\mathbb{K}g \subset \mathrm{GL}_d(\hat{A})$, there exists a real number $\beta'_g > \beta_g$ such that the isomorphism $\tilde{\xi}_g$ sends $\tilde{Y}_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D},g'}$ to $\tilde{Y}_{g\mathbb{K}g^{-1},\bullet}^{(\alpha-\beta_g),\mathcal{D},g'g} \subset \tilde{X}_{g\mathbb{K}g^{-1},\bullet}$ for $\alpha > \beta'_g$, for any $f \in \mathrm{Flag}_{\mathcal{D}}$ and for any $g' \in \mathrm{GL}_d(\mathbb{A}^\infty)$. Hence, as a smooth $\mathrm{GL}_d(\mathbb{A}^\infty)$ -module, $\varinjlim_{\mathbb{K}} \varinjlim_{\alpha} H^*(X_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}, \mathbb{Q})$ is isomorphic to $\mathrm{Ind}_{P_{\mathcal{D}}(\mathbb{A}^\infty)}^{\mathrm{GL}_d(\mathbb{A}^\infty)} \varinjlim_{\mathbb{K}} \varinjlim_{\alpha} H^*(Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}, \mathbb{Q})$.

Lemma 35. *For any $g \in \mathrm{GL}_d(\mathbb{A}^\infty)$, the simplicial complex $\tilde{X}_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D},f_0} \cap (\{g\mathbb{K}\} \times \mathcal{BT}_\bullet)$ is non-empty and contractible.*

Proof. Since $\tilde{X}_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D},f_0} \cap (\{g\mathbb{K}\} \times \mathcal{BT}_\bullet)$ is isomorphic to $\tilde{X}_{\mathrm{GL}_d(\hat{A}),\bullet}^{(\alpha),\mathcal{D},f_0} \cap (\{g\mathrm{GL}_d(\hat{A})\} \times \mathcal{BT}_\bullet)$, we may assume that $\mathbb{K} = \mathrm{GL}_d(\hat{A})$. We set $X = \tilde{X}_{\mathrm{GL}_d(\hat{A}),\bullet}^{(\alpha),\mathcal{D},f_0} \cap (\{g\mathrm{GL}_d(\hat{A})\} \times \mathcal{BT}_{\mathrm{GL}_d(\hat{A})})$.

We proceed by induction on d , in a manner similar to that in the proof of Theorem 4.1 of [Gr]. Let $i \in \mathcal{D}$ be the minimal element and set $d' = d - i$. We define the subset $\mathcal{D}' \subset \{1, \dots, d' - 1\}$ as $\mathcal{D}' = \{i' - i \mid i' \in \mathcal{D}, i' \neq i\}$. We define $f'_0 \in \mathrm{Flag}_{\mathcal{D}'}$ as the image of the flag f_0 in $F^{\oplus d}$ with respect to the projection $F^{\oplus d} \twoheadrightarrow F^{\oplus d} / (F^{\oplus i} \oplus \{0\}^{\oplus d'}) \cong F^{\oplus d'}$. Take an element $g' \in \mathrm{GL}_{d'}(\mathbb{A}^\infty)$ such that the quotient $\hat{A}^{\oplus d} g'^{-1} / (\hat{A}^{\oplus d} g'^{-1} \cap (\mathbb{A}^{\infty \oplus i} \oplus \{0\}^{\oplus d'}))$ equals $\hat{A}^{\oplus d'} g'^{-1}$ as an \hat{A} -lattice of $\mathbb{A}^{\infty \oplus d'}$. We set $X' = \tilde{X}_{\mathrm{GL}_{d'},\mathrm{GL}_{d'}(\hat{A}),\bullet}^{(\alpha),\mathcal{D}',f'_0} \cap (\{g'\mathrm{GL}_{d'}(\hat{A})\} \times \mathcal{BT}_{\mathrm{GL}_{d'}(\hat{A})})$ if \mathcal{D}' is non-empty. Otherwise we set $X' = \tilde{X}_{\mathrm{GL}_{d'},\mathrm{GL}_{d'}(\hat{A}),\bullet} \cap (\{g'\mathrm{GL}_{d'}(\hat{A})\} \times \mathcal{BT}_{\mathrm{GL}_{d'}(\hat{A})})$. By induction hypothesis, $|X'|$ is contractible. There is a canonical morphism $h : X \rightarrow X'$ which sends an \mathcal{O}_C -submodule $\mathcal{F}[g, L_\infty]$ of $\eta_* F^{\oplus d}$ to the \mathcal{O}_C -submodule $\mathcal{F}[g, L_\infty] / \mathcal{F}[g, L_\infty]_{(i)}$ of $\eta_* F^{\oplus d'}$. Let $\epsilon : \mathrm{Vert}(X) \rightarrow \mathbb{Z}$ and $\epsilon' : \mathrm{Vert}(X') \rightarrow \mathbb{Z}$ denote the maps which send a locally free \mathcal{O}_C -module \mathcal{F} to the integer $[p_{\mathcal{F}}(1) / \deg(\infty)]$. We fix an \mathcal{O}_C -submodule \mathcal{F}_0 of $\eta_* F^{\oplus d}$ whose equivalence class belongs to X . By twisting \mathcal{F}_0 by some power of $\mathcal{O}_C(\infty)$ if necessary, we may assume that $p_{\mathcal{F}_0}(i) - p_{\mathcal{F}_0}(i-1) > \alpha$. We fix a splitting $\mathcal{F}_0 = \mathcal{F}_{0,(i)} \oplus \mathcal{F}'_0$. This splitting induces an isomorphism $\varphi : \eta_* \eta^* \mathcal{F}'_0 \cong \eta_* F^{\oplus d}$. Let $h' : X' \rightarrow X$ denote the morphism which sends an \mathcal{O}_C -submodule \mathcal{F}' of $\eta_* \eta^* F^{\oplus d'}$ to the \mathcal{O}_C -submodule $\mathcal{F}_{0,(i)}(\epsilon'(\mathcal{F}')\infty) \oplus \varphi^{-1}(\mathcal{F}')$ of $\eta_* F^{\oplus d}$. For each $n \in \mathbb{Z}$, define a morphism $G_n : X \rightarrow X$ by sending an \mathcal{O}_C -submodule \mathcal{F} of $\eta_* \eta^* F^{\oplus d}$ to the \mathcal{O}_C -submodule $\mathcal{F}_{0,(i)}((n + \epsilon(\mathcal{F}))\infty) + \mathcal{F}$ of $\eta_* F^{\oplus d}$. Then the argument in [Gr, p. 85–86] shows that f and $|h'| \circ |h| \circ f$ are homotopic for any map $f : Z \rightarrow |X|$ from a compact space Z to $|X|$. Since the map $|h'| \circ |h| \circ f$ factors through the contractible space $|X'|$, f is null-homotopic. Hence $|X|$ is contractible. \square

Proof of Proposition 31. For any simplex σ in $\tilde{X}_{\mathbb{K},\bullet}$, the isotropy group $\Gamma_\sigma \subset \mathrm{GL}_d(F)$ is finite, as remarked in Section 4.1. Hence by Lemma 35, both $H^*(Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D},g}, \mathbb{Q})$ and $H^*(Y_{\mathbb{K},\bullet}^{(\alpha'),\mathcal{D},g}, \mathbb{Q})$ are canonically isomorphic to the same group $H^*(P_{\mathcal{D}}(F), \mathrm{Map}(P_{\mathcal{D}}(\mathbb{A})g / (g^{-1}P_{\mathcal{D}}(\mathbb{A}^\infty)g \cap \mathbb{K}), \mathbb{Q}))$ for any non-empty subset $\mathcal{D} \subset \{1, \dots, d-1\}$ and for $g \in \mathrm{GL}_d(\mathbb{A}^\infty)$. This shows that $H^*(X_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}, \mathbb{Q}) \rightarrow H^*(X_{\mathbb{K},\bullet}^{(\alpha'),\mathcal{D}}, \mathbb{Q})$ is an isomorphism. Since the homomorphisms between the E_1 -terms of the spectral sequences (7.2) for α and for α' is an isomorphism, $H^*(X_{\mathbb{K},\bullet}^{(\alpha)}, \mathbb{Q}) \rightarrow H^*(X_{\mathbb{K},\bullet}^{(\alpha')}, \mathbb{Q})$ is an isomorphism. \square

7.6. Proof of Proposition 33. For $j = 1, \dots, r$, let $\mathbb{K}_j \subset \mathrm{GL}_{d_i}(\mathbb{A}^\infty)$ denote the image of $\mathbb{K} \cap P_{\mathcal{D}}(\mathbb{A}^\infty)$ by the composition $P_{\mathcal{D}}(\mathbb{A}^\infty) \rightarrow M_{\mathcal{D}}(\mathbb{A}^\infty) \rightarrow \mathrm{GL}_{d_i}(\mathbb{A}^\infty)$.

We define the continuous map $\tilde{\pi}_{\mathcal{D},j} : |\tilde{Y}_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}| \rightarrow |\tilde{X}_{\mathrm{GL}_{d_j},\mathbb{K}_j,\bullet}|$ of topological spaces in the following way. Let σ be an i -simplex in $\tilde{Y}_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}$. Take a chain $\dots \supseteq \mathcal{F}_{-1} \supseteq \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \dots$ of \mathcal{O}_C -modules representing σ . For $l \in \mathbb{Z}$ we set $\mathcal{F}_{l,j} = \mathcal{F}_{l,(i_j)} / \mathcal{F}_{l,(i_{j-1})}$, which is an \mathcal{O}_C -submodule of $\eta_* F^{\oplus d_j}$. We set $S_j = \{l \in \mathbb{Z} \mid \mathcal{F}_{l,j} \neq \mathcal{F}_{l+1,j}\}$. Define the map $\psi_j : \mathbb{Z} \rightarrow S_j$ as $\psi_j(l) = \min\{l' \geq l \mid l' \in S_j\}$. Take an order-preserving bijection $\varphi_j : S_j \xrightarrow{\cong} \mathbb{Z}$. For $l \in \mathbb{Z}$ set $\mathcal{F}'_l = \mathcal{F}_{\varphi_j^{-1}(l),j}$. Then the chain $\dots \supseteq \mathcal{F}'_{-1} \supseteq \mathcal{F}'_0 \supseteq \mathcal{F}'_1 \supseteq \dots$ defines a simplex σ' in $\tilde{X}_{\mathrm{GL}_{d_j},\mathbb{K}_j,\bullet}$. We define a continuous map $|\sigma| \rightarrow |\sigma'|$ as the affine map sending the vertex of σ corresponding to \mathcal{F}_l to the vertex of σ' corresponding to $\mathcal{F}'_{\varphi_j \circ \psi_j(l)}$. Gluing these maps, we obtain a

continuous map $\tilde{\pi}_{\mathcal{D},j} : |\tilde{Y}_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}| \rightarrow |\tilde{X}_{\mathrm{GL}_{d_j},\mathbb{K}_j,\bullet}|$. We set $\tilde{\pi}_{\mathcal{D}} = (\tilde{\pi}_{\mathcal{D},1}, \dots, \tilde{\pi}_{\mathcal{D},r}) : |\tilde{Y}_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}| \rightarrow \prod_{j=1}^r |\tilde{X}_{\mathrm{GL}_{d_j},\mathbb{K}_j,\bullet}|$. This continuous map descends to the continuous map $\pi_{\mathcal{D}} : |Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}| \rightarrow \prod_{j=1}^r |X_{\mathrm{GL}_{d_j},\mathbb{K}_j,\bullet}|$.

If $g \in P_{\mathcal{D}}(\mathbb{A}^\infty)$ and $g^{-1}\mathbb{K}g \subset \mathrm{GL}_d(\hat{A})$, then the isomorphism $\xi_g : X_{\mathbb{K},\bullet} \xrightarrow{\cong} X_{g^{-1}\mathbb{K}g,\bullet}$ sends $Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}$ inside $Y_{g^{-1}\mathbb{K}g,\bullet}^{(\alpha-\beta_g),\mathcal{D}}$. If we denote by (g_1, \dots, g_r) the image of g in $M_{\mathcal{D}}(\mathbb{A}^\infty) = \prod_{j=1}^r \mathrm{GL}_{d_j}(\mathbb{A}^\infty)$, then the diagram

$$\begin{array}{ccc} |Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}| & \xrightarrow{\xi_g} & |Y_{g^{-1}\mathbb{K}g,\bullet}^{(\alpha-\beta_g),\mathcal{D}}| \\ \pi_{\mathcal{D}} \downarrow & & \downarrow \pi_{\mathcal{D}} \\ \prod_{j=1}^r |X_{\mathrm{GL}_{d_j},\mathbb{K}_j,\bullet}| & \xrightarrow{(\xi_{g_1}, \dots, \xi_{g_r})} & \prod_{j=1}^r |X_{\mathrm{GL}_{d_j},g_j^{-1}\mathbb{K}_jg_j,\bullet}| \end{array}$$

is commutative.

With the notations as above, suppose that the open compact subgroup $\mathbb{K} \subset \mathrm{GL}_d(\mathbb{A}^\infty)$ has the following property.

(7.4) the homomorphism $P_{\mathcal{D}}(\mathbb{A}^\infty) \cap \mathbb{K} \rightarrow \mathbb{K}_1 \times \dots \times \mathbb{K}_r$ is surjective.

For a simplicial complex X , we set $I_X = \mathrm{Map}(\pi_0(X), \mathbb{Q})$, where $\pi_0(X)$ is the set of the connected components of X . Let us consider the following commutative diagram.

$$(7.5) \quad \begin{array}{ccc} H^*(M_{\mathcal{D}}(F), \mathrm{Map}(\prod_{j=1}^r \pi_0(X_{\mathrm{GL}_{d_j},\mathbb{K}_j,\bullet}), \mathbb{Q})) & \longrightarrow & H^*(P_{\mathcal{D}}(F), I_{Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}}) \\ \downarrow & & \downarrow \\ H_{M_{\mathcal{D}}(F)}^*(\prod_{j=1}^r |\tilde{X}_{\mathrm{GL}_{d_j},\mathbb{K}_j,\bullet}|, \mathbb{Q}) & \longrightarrow & H_{P_{\mathcal{D}}(F)}^*(|Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}|, \mathbb{Q}) \\ \uparrow & & \uparrow \\ H^*(\prod_{j=1}^r |X_{\mathrm{GL}_{d_j},\mathbb{K}_j,\bullet}|, \mathbb{Q}) & \longrightarrow & H^*(|Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}|, \mathbb{Q}). \end{array}$$

Here $H_{M_{\mathcal{D}}(F)}^*$ and $H_{P_{\mathcal{D}}(F)}^*$ denote the equivariant cohomology groups.

Proposition 36. *All homomorphisms in the above diagram (7.5) are isomorphisms.*

Proof. We prove that the upper horizontal arrow and the four vertical arrows are isomorphisms.

First we consider the upper horizontal arrow.

Lemma 37. *For $q \geq 1$, the group $H^q(N_{\mathcal{D}}(F), I_{Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}})$ is zero.*

Proof of Lemma 37. For each $x \in N_{\mathcal{D}}(F) \setminus \pi_0(Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}})$, take a lift $\tilde{x} \in Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}$ of x and let $N_x \subset N_{\mathcal{D}}(F)$ denote the stabilizer of \tilde{x} . Then the group $H^*(N_{\mathcal{D}}(F), I_{Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}})$ is isomorphic to the direct product

$$\prod_{x \in N_{\mathcal{D}}(F) \setminus \pi_0(Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}})} H^*(N_x, \mathbb{Q}).$$

We note that the group $N_{\mathcal{D}}(F)$ is a union $N_{\mathcal{D}}(F) = \bigcup_i U_i$ of finite subgroups of p -power order where p is the characteristic of F . (This follows easily from [KeWe, p.2, 1.A.2 Lemma] or from [KeWe, p.60, 1.L.1 Theorem].) Hence $N_x = \bigcup_i (U_i \cap N_x)$. Therefore, for an N_x -module M , $H^j(N_x, M) = 0$ for $j \geq 1$ if the projective system $(H^0(U_i \cap N_x, M))_i$ satisfies the Mittag-Leffler condition. In particular we have $H^j(N_x, \mathbb{Q}) = 0$ for $j \geq 1$. Hence the claim follows. \square

We note that $\pi_0(X_{\mathrm{GL}_{d_j},\mathbb{K}_j,\bullet})$ is canonically isomorphic to $\mathrm{GL}_{d_j}(\mathbb{A}^\infty)/\mathbb{K}_j$ for $j = 1, \dots, r$, and Lemma 35 implies that $I_{Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}}$ is canonically isomorphic to $\mathrm{Map}(P_{\mathcal{D}}(\mathbb{A}^\infty)/(P_{\mathcal{D}}(\mathbb{A}^\infty) \cap \mathbb{K}), \mathbb{Q})$. Since $N_{\mathcal{D}}(F)$ is dense in $N_{\mathcal{D}}(\mathbb{A}^\infty)$, the group $H^0(N_{\mathcal{D}}(F), I_{Y_{\mathbb{K},\bullet}^{(\alpha),\mathcal{D}}})$ is canonically isomorphic to the group $\mathrm{Map}(M_{\mathcal{D}}(\mathbb{A}^\infty)/\prod_{j=1}^r \mathbb{K}_j, \mathbb{Q})$. Hence the upper horizontal arrow of the diagram (7.5) is an isomorphism.

Next we consider the two vertical arrows. Each connected component of $\tilde{X}_{\mathrm{GL}_{d_j},g_j^{-1}\mathbb{K}_jg_j,\bullet}$ is contractible since it is isomorphic to the Bruhat-Tits building for GL_{d_j} . Recall that the simplicial complex $X_{\mathrm{GL}_{d_j},g_j^{-1}\mathbb{K}_jg_j,\bullet}$ is the quotient of $\tilde{X}_{\mathrm{GL}_{d_j},g_j^{-1}\mathbb{K}_jg_j,\bullet}$ by the action of $\mathrm{GL}_{d_j}(F)$. For any simplex σ in $\tilde{X}_{\mathrm{GL}_{d_j},g_j^{-1}\mathbb{K}_jg_j,\bullet}$, the isotropy group $\Gamma_\sigma \subset \mathrm{GL}_{d_j}(F)$ of σ is finite, as remarked in Section 4.1. Hence the left two vertical arrows in the diagram (7.5) are isomorphisms. Similarly, bijectivity of the two right vertical arrows in the diagram (7.5) follows from Lemma 35. Thus we have a proof of Proposition 36. \square

Proof of Proposition 33. Let us consider the lower horizontal arrow in the diagram (7.5). By Proposition 36 it is an isomorphism. We note that the compact open subgroups $\mathbb{K} \subset \mathrm{GL}_d(\widehat{A})$ with property (7.4) form a cofinal subsystem of the inductive system of all open compact subgroups of $\mathrm{GL}_d(\mathbb{A}^\infty)$. Therefore, passing to the inductive limits with respect to α and \mathbb{K} with property (7.4), we have $\varinjlim_{\mathbb{K}} \varinjlim_{\alpha} H^*(Y_{\mathbb{K}, \bullet}^{(\alpha), \mathcal{D}}, \mathbb{Q}) \cong \bigotimes_{j=1}^r \varinjlim_{\mathbb{K}_j} H^*(X_{\mathrm{GL}_{d_j}, \mathbb{K}_j, \bullet}, \mathbb{Q})$ as desired. \square

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